$d(h(a), a) < \varepsilon(a)$ and $h(A) \subset \operatorname{rint} Q \times s^{\omega}$. It is possible to verify that $h: A \subsetneq ((-1, 1) \times s)^{\omega}$ is an inclusion that is identical on B and h(A) is a closed set in s × s^{ω}. Since $h(A) = \cup\{h(A) \cap X_n | n \geqslant 1\}$ is the countable union of Z-sets, h(A) is itself a Z-set. This means that h is a Z-inclusion that is $\mathcal{U}$-close to id|A. The lemma is proved.

Note that if $B \subset A \cap (\sigma \times s^{\omega})$, then $h(A) \subset \sigma \times s^{\omega}$; thus, it has been shown that the space σ × s is a $\mathfrak{X}$-absorbing set in s × s.

In conclusion, the author would like to express kind appreciation to M. M. Zarichniy for attention in the work and the useful discussion of results.

LITERATURE CITED

1. J. Milnor, "Construction of universal bundles. II," Ann. Math., 63, No. 3, 430-436 (1956).
2. H. Torunczyk, "Characterizing Hilbert space topology," Fund. Math., 101, 93-110 (1978).
3. C. Bessaga and A. Pelczynski, Selected Topics in Infinite-Dimensional Topology,' PWN, Warsaw (1975).
4. D. Huezmoller, Stratified Spaces [Russian translation], Mir, Moscow (1970).
5. A. Dold, "Partitions of unity in the theory of fibrations," Ann. Math., 78, 223-255 (1963).
6. T. O. Banakh, "On the continuation of pseudometrics onto the space of probabilistic measures," Vestn. L'vovsk. Univ. Ser. Mat.-Mekh., 30, 46-48 (1988).
7. J. Dugunji, "An extention of Tietze's theorem." Pacific J. Math., 1, 353-367 (1951).
8. T. Dobrowolski and H. Torunczyk, "Separable complete ANR's admitting a group structure are Hilbert manifolds," Topology Appl., 12, 229-235 (1981).

JOINTLY DISSIPATIVE OPERATORS AND THEIR APPLICATIONS

V. I. Verbitskii and A. N. Gorban'

UDC 517.9

The jointly dissipative operators introduced by Verbitskii and Gorban' [1] find application in the analytis of dynamical properties of nonlinear systems of ordinary differential equations [1, 2] and in some applications (chemical kinetics [1, 2], numerical analysis). In the present paper we discuss the properties of jointly dissipative operators and some of their applications.

Let E be an n-dimensional real or complex linear space, and let L(E) be the space of linear operators in E. Let us introduce a norm $\|\cdot\|$ on E and the corresponding norm in L(E). An operator $A \in L(E)$ is said to be dissipative [4] if $\|\exp(tA)\| \leqslant 1$ for all $t \geqslant 0$.

Definition 1. An operator $A \in L(E)$ is said to be roughly dissipative if there is an ε > 0 such that $\|\exp(tA)\| \leqslant \exp(-\varepsilon t)$ for all $t \geqslant 0$.

The dissipativity of an operator is determined by the sign of its Lozinskii logarithmic norm [5]:

$$\gamma(A) = \lim_{h \to +0} \frac{\|I + hA\| - 1}{h},$$

where I is the identity operator. It is easy to show that an operator is dissipative [resp., roughly dissipative] with respect to a given norm if the corresponding Lozinskii norm is nonpositive [resp., negative]. Since for Euclidean and polyhedral norms the logarithmic norm of an operator is relatively easy to express through its matrix elements [5, 6], verifying dissipativity [resp., rough dissipativity] of an operator in such norms is not a complicated task.

It is readily seen that a norm with respect to which a given operator A is dissipative exists if and only if the system

$$\frac{dx}{dt} = Ax \qquad (1)$$

Krasnoyarsk. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 33, No. 1, pp. 26-31, January-February, 1992. Original article submitted April 17, 1990.

is Lyapunov-stable. As is known [7], in order for this to be the case it is necessary and sufficient that the spectrum of A lie in the closed left half-plane and the set of Jordan blocks corresponding to purely imaginary eigenvalues (including the zero eigenvalue) be diagonal. For the existence of a norm with respect to which the operator A will be roughly dissipative it is necessary and sufficient that the system (1) be asymptotically stable, i.e., that the matrix of A be stable (i.e., that the spectrum of A Lie in the open left half-plane). This condition is verified constructively by means of the Routh–Hurwitz criterion [7].

The notions of dissipativity and rough dissipativity can be interpreted geometrically as follows. For each point x of the unit sphere ($\|x\| = 1$) one defines an open cone Q_x, directed "strictly inward" the unit ball: $y \in Q_x$ if and only if there exists an $\varepsilon_0 > 0$ such that $\|x + \varepsilon y\| < 1$ for all $\varepsilon \in (0; \varepsilon_0)$. In [1] it is proved that an operator $A \in L(E)$ is roughly-dissipative in a given norm if and only if for any point x of the unit sphere $Ax \in Q_x$, and the operator A is dissipative if and only if for any point x of the unit sphere Ax belongs to $\bar{Q}_x$ (the closure of Q_x).

As a consequence of this result the set of all operators that are roughly dissipative with respect to a norm $\|\cdot\|$ can be identified with the interior of the set of all operators that are dissipative with respect to $\|\cdot\|$. Therefore, the rough dissipativity property is preserved under small perturbations of operators.

In [1] it is also established that the dissipative [resp., roughly dissipative] operators in a given norm form a closed [resp., open] cone. This cone is uniquely determined by the norm up to a scalar factor.

Let us now give our main

Definition 2. A family of operators $\{A_\alpha\}$ is said to be jointly dissipative [resp., jointly roughly dissipative] if there exists a norm with respect to which all operators A_α are dissipative [resp., roughly dissipative].

The need for this definition can be motivated, for example, by the fact that operators that are dissipative, each in its own norm, are not necessarily jointly dissipative. Indeed, consider the operators given by the matrices

$$A_1 = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix}.$$

Each of them is roughly-dissipative in its own norm, because the matrices A_1 and A_2 are stable. However,

$$A_1 + A_2 = \begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}.$$

The spectrum of this matrix contains the point $\lambda = 1$, which does not belong to the closed left half-plane. Thus, the operator $(A_1 + A_2)$ is not dissipative, whatever the norm, and therefore the operators A_1 and A_2 are not jointly dissipative.

The problem of finding necessary and sufficient conditions for the joint dissipativity of an arbitrary family of operators is apparently extremely difficult. Some sufficient conditions can be obtained by imposing various constraints on the operators. The sufficient conditions of joint dissipativity given below are based on algebraic properties of the given family. Recall [8] that a family of matrices generates a solvable Lie algebra if and only if its members are simultaneously reducible to triangular form in some complex basis.

THEOREM 1. Suppose the family $\{A_\alpha\}$ is compact, generates a solvable Lie algebra, and all matrices A_α are stable. Then $\{A_\alpha\}$ is jointly roughly dissipative.

We first carry out the proof for the case of a complex space E. Consider the matrices A_α in basis $\{e_k\}_{k=1}^{n}$ in which they are triangular. Suppose each matrix A_α has the form

$$A_\alpha = \begin{pmatrix} \lambda_1^\alpha & 0 & 0 \dots & 0 \\ \mu_{21}^\alpha & \lambda_2^\alpha & 0 \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n1}^\alpha & \mu_{n2}^\alpha & \dots \mu_{n,n-1}^\alpha & \lambda_n^\alpha \end{pmatrix}. \tag{2}$$

Let us show that there exists a set of positive numbers $\{a_k\}_{n=1}^{n}$ such that all A_α are roughly dissipative in the norm

$$\|z\| = \max_{1\leqslant k\leqslant n} \{|z_k|/a_k\}, \tag{3}$$

where z_k is the k-th coordinate of the vector z in the basis $\{e_k\}$. Clearly, a ball in this norm is a polycylinder

$$|z_k| \leqslant a_k \quad (k=1, \ldots, n). \tag{4}$$

The norm (3) coincide with the ℓ^∞-norm with respect to the basis $\{a_k e_k\}$. Using the explicit form of the Lozanskii norm for the ℓ^∞-norm [6], we infer that the conditions of rough dissipativity of the matrices (2) with respect to the norm (3) have the form

$$\sum_{j=1}^{k-1} a_j |\mu_{kj}^\alpha| + a_k \cdot \operatorname{Re}\lambda_k < 0 \ (k=2, \ldots, n); \quad a_1 > 0. \tag{5}$$

Set $\mu = \sup\limits_{\alpha,k\neq l} |\mu_{kl}^\alpha|$; $\lambda = -\sup\limits_{\alpha,k} \operatorname{Re}\lambda_k^\alpha$. From the hypotheses of the theorem it follows that $0 < \lambda < +\infty$ and $0 < \mu < +\infty$. In order for (5) to be satisfied is suffices that the following inequalities hold:

$$(a_1 + \ldots + a_{k-1})\mu < a_k\lambda \ (k=2, \ldots, n); \quad a_1 > 0. \tag{6}$$

Let us show the compatibility of system (6). Set $a_1 = 1$. Then choose the remaining a_k so that

$$a_2 > \mu/\lambda;\ a_3 > (1+a_2)\mu/\lambda;\ \ldots;\ a_n > (1+a_2+\ldots+a_{n-1})\mu/\lambda. \tag{7}$$

Then inequalities (6) hold, and the theorem is established in the case where E is complex.

Now suppose E is a real space. In this case we consider the intersection of the polycylinder (4) with the original space E. We obtain a ball in a norm with respect to which the family $\{A_\alpha\}$ is jointly roughly dissipative. This completes the proof of the theorem.

If instead of roughly dissipative we simply consider dissipative operators, the analogue of Theorem 1 is no longer true starting with the real dimension 4 (respectively, complex dimension 2). A counterexample is constructed as follows. In $\mathbf{C}^2$ consider the two matrices

$$A_1 = \begin{pmatrix} i & 1 \\ 0 & 2i \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2i & 1 \\ 0 & i \end{pmatrix}.$$

Each of them is dissipative in its own norm (their spectra lie in the closed left half-plane, and the matrices are simultaneously reducible to diagonal form). A finite family is compact. Moreover, A_1 and A_2 generate a solvable Lie algebra. However,

$$A_1 + A_2 = \begin{pmatrix} 3i & 2 \\ 0 & 3i \end{pmatrix}.$$

The unique eigenvalue of this matrix is purely imaginary, and it cannot be reduced to diagonal form. Therefore, this matrix is not dissipative, whatever the norm, i.e., the matrices A_1 and A_2 are not jointly dissipative. To produce a counterexample over the real field, it remain to realify the matrices:

$$A_1^{\mathbf{R}} = \begin{Vmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{Vmatrix}, \quad A_2^{\mathbf{R}} = \begin{Vmatrix} 0 & -2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{Vmatrix}.$$

In order that the assertion on joint dissipativity remain in force for operators that are not roughly dissipative, it suffices to strengthen the compactness requirement to finiteness, and that of solvability to nilpotency. It is known [8] that to any operator A in a space E there corresponds the operator ad A in L(E), acting according to the rule

$$(\operatorname{ad} A)B = AB - BA.$$

A family $\{A_\alpha\}$ generates a nilpotent Lie algebra if and only if there exists an $m \in \mathbf{N}$ such that for any collection $\{A_{\alpha k}\}_{k=1}^{m}$repetition of elements is allowed) and any α one has that

$$\prod_{k=1}^{m} (\operatorname{ad} A_{\alpha k})\, A_\alpha = 0. \tag{8}$$

A nilpotent Lie algebra is always solvable. A commutative algebra is obviously nilpotent (for it m = 1), and hence solvable.

THEOREM 2. Suppose the family $\{A_\alpha\}$ is finite, generates a nilpotent Lie algebra, and for each operator A_α there exists a norm with respect to which A_α is dissipative. Then the family $\{A_\alpha\}$ is jointly dissipative.

Sketch of the Proof. With no loss in generality one can assume that none of the operators A_α is scalar (if $A = \lambda I$, where $\operatorname{Re}\lambda \leqslant 0$, then A is dissipative in any norm) and that for at least one of them (call it A_1) the spectrum contains purely imaginary eigenvalues (otherwise we would be in the hypotheses of Theorem 1).

The proof is carried out by induction on the dimension of the space. In dimension 1 the assertion of the theorem is trivial. Let λ be a purely imaginary eigenvalue of A_1, and let E' be the corresponding eigensubspace (in view of the fact that the corresponding Jordan blocks are diagonal, E' coincides with all root subspaces corresponding to λ). Let E" be the sum of root subspaces corresponding to all other eigenvalues of A. Clearly, E decomposes into the direct sum of E' and E". From the fact that the Lie algebra generated by $\{A_\alpha\}$ is nilpotent one can infer that E' and E" are invariant under all operators A_α. Moreover, $\dim E' < n$, $\dim E'' < n$; otherwise, A is scalar. To complete the proof it remains to use a simple fact: if a space E is decomposed into a direct sum of subspaces E_i that are invariant under the family $\{A_\alpha\}$ and the restriction of $\{A_\alpha\}$ to any E_i is jointly dissipative [resp., jointly roughly dissipative], then $\{A_\alpha\}$ is itself jointly dissipative [resp., jointly roughly dissipative].

A particular consequence of Theorems 1 and 2 is that a compact [resp., finite] commutative family of operators, each of which is roughly dissipative [resp., dissipative] in its own norm is jointly roughly dissipative [resp., jointly dissipative].

To conclude the paper let us consider some applications of joint dissipativity to the dynamics of nonlinear systems.

We shall consider nonlinear, generally speaking nonautonomous systems of the form

$$\frac{dx}{dt} = f(x, t), \tag{9}$$

where f is an $\mathbf{R}^n$ -valued map of class $C^{1,0}(U \times S)$ (i.e., smooth in x for $x \in U$, with U a domain in $\mathbf{R}^n$, and continuous in t for $t \in S$, with S either a ray $[\theta; +\infty)$, with $\theta \in \mathbf{R}$, or the full real line $\mathbf{R}$). Let $B \subset U$ be a convex forward-invariant set of system (9).

Definition 3 [1]. System (9) is said to be contractive [resp., exponentially contractive] on the set B with respect to the norm $\|\cdot\|$ if there exists an $\varepsilon \geqslant 0$ such that for any two solutions $x_1(t)$, $x_2(t)$ of (9) with initial conditions $x_1(t_0) \in B$, $x_2(t_0) \in B$ $(t_0 \in S)$ the inequality

$$\|x_1(t) - x_2(t)\| \leqslant \exp(-\varepsilon(t-s))\|x_1(s) - x_2(s)\| \tag{10}$$

holds for all $t \geqslant s \geqslant t_0$.

Clearly, a system that is contractive [resp., exponentially contractive] on a set B with respect to some norm is stable [resp., exponentially stable] in the wide sense [9] with respect to B.

We shall denote by J(x, t) the restriction of the derivative of the map f with respect to x to the subspace parallel to the affine hull of B.

THEOREM 3. If order that the system (9) be contractive [resp., exponentially contractive] on B in the norm $\|\cdot\|$ it is necessary and sufficient that there exist a $\nu \geqslant 0$ [resp., $\nu > 0$] such that

$$\gamma(J(x, t)) \leqslant -\nu \tag{11}$$

for all $x \in B$, $t \in S$ (where γ is the Lozanskii norm corresponding to $\|\cdot\|$). If this is the case, then inequality (10) holds with $\varepsilon = \nu$.

From Theorem 3 it follows that for the existence of a norm with respect to which the system (9) will be contractive [resp., exponentially contractive] it is necessary and sufficient that the operators $J(x, t)$ $(x \in B,\ t \in S)$ be jointly dissipative [resp., that there exists a $\nu > 0$ such that the operators $\{J(x, t) + \nu I\}$ are jointly dissipative; if B is compact and the system is autonomous, then the last condition means that the operators J(x, t) are jointly roughly dissipative].

Sketch of the Proof. One shows that inequality (10) holds for system (9) if and only if it holds for the solutions of any variational system corresponding to (9). After this, the necessity and sufficiency of inequality (11) is established by resorting to the following estimate of the solutions of the system $dx/dt = A(t)x$:

$$\frac{d}{dt}\|x(t)\| \leqslant \gamma(A(t))\|x(t)\|.$$

In this way, the problem of contractivity has been reduced to a problem of joint dissipativity. Theorem 3 can be used, for example, to investigate the stability of systems of chemical kinetics [1, 2].

Let us consider the autonomous system

$$\frac{dx}{dt} = f(x), \tag{12}$$

where f(x) is a twice-differentiable map ($U \subset \mathbf{R}^n \to \mathbf{R}^n$). Let $B \subset U$ be a convex, compact, forward-invariant set for (12), let J(x) denote the Jacobi matrix of f, let T_t, $t \geqslant 0$, be the time-t map of the phase flow of system (12). Let $M = \bigcap_{t\geqslant 0} T_t B$ be the maximal attractor of system (12) in B. Define a linear operator $\mathbf{R}^n$ in the k-th exterior power of $\mathscr{D}_{xk}$ as follows: $\mathscr{D}_{xk}(y_1 \wedge \ldots \wedge y_k) = (J_x y_1) \wedge y_2 \wedge \ldots \wedge y_k + y_1 \wedge (J_x y_2) \wedge \ldots \wedge y_k + \ldots + y_1 \wedge \ldots \wedge y_{k-1} \wedge (J_x y_k)$.

THEOREM 4. If the family $\{\mathscr{D}_{xk}\ (x \in B)\}$ is jointly roughly dissipative, then the Hausdorff dimension of M does not exceed k [10].

Theorem 4 represents a generalization of a result of Il'yashenko and Chetaev [10, 11] (the conditions of the Il'yashenko–Chetaev theorem actually reduce to the joint dissipativity of the operators $\mathscr{D}_{xk}$ is a concrete Euclidean norm). The proof is carried out following the same scheme as in [10], Theorem 4 allow one, under wider hypotheses than in [10, 11], to estimate the Hausdorff dimension of attractors of autonomous systems.

In conclusion let us point out that the utilization of joint dissipativity of operators allows one to generalize a number of known theorems on stability [12, 13].

LITERATURE CITED

1. V. I. Verbitskii and A. N. Gorban', "Thermodynamic constraints and quasithermodynamicity conditions in chemical kinetics," in: Mathematical Problems of Chemical Kinetics [in Russian], Nauka, Novosibirsk (1989), pp. 42-83.
2. V. I. Verbitskii, "Jointly dissipative operators and their applications in the investigation of nonlinear system," Candidate's Dissertation, Vychisl. Tsentr Sib. Otdel. Akad. Nauk SSSR, Krasnoyarsk (1987).
3. V. I. Verbitskii, A. N. Gorban', and G. Sh. Utyubaev, "The Moore effect in interval spaces," Dokl. Akad. Nauk SSSR, 304, No. 1, 17-21 (1989).
4. G. R. Belitskii and Yu. I. Lyubich, Matrix Norms and Their Applications [in Russian], Naukova Dumka, Kiev (1984).
5. S. M. Lozinskii, "Estimation of the errors of numerical integration of ordinary differential equations," Izv. Vyssh. Uchebn. Zaved., Ser. Mat., No. 5, 52-90 (1958).
6. B. V. Bylov, R. E. Vinograd, D. M. Grobman, and V. V. Nemytskii, The Theory of Lyapunov Exponents [in Russian], Nauka, Moscow (1966).
7. B. P. Demidovich, Lectures on the Mathematical Theory of Stability [in Russian], Nauka, Moscow (1967).
8. I. Kaplansky, Lie Algebras and Locally Compact Groups, The Univ. of Chicago Press, Chicago (1971).
9. L. Cesari, Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, Third edition, Springer, New York (1971).
10. Yu. S. Il'yashenko, "Weakly contractive systems and attractors of Galerkin approximations of the Navier–Stokes equations on the two-dimensional torus," Usp. Mekh., 5, Nos. 1-2, 32-63 (1983).
11. Yu. S. Il'yashenko and A. N. Chetaev, "On the dimension of attractors for one class of dissipative systems," Prikl. Mat. Mekh., 46, No. 3, 374-381 (1982).
12. N. N. Krasovskii, "Sufficient conditions for the stability of solutions of a system of nonlinear differential equations," Dokl. Akad. Nauk SSSR, 98, No. 6, 901-904 (1954).
13. V. M. Cheresiz, "Stable and conditionally stable almost-periodic solutions of V-monotone systems," Sib. Mat. Zh., 15, No. 1, 162-176 (1974).