8 Hydrodynamics From Grad's Equations: What Can We Learn From Exact Solutions?

A detailed treatment of the classical Chapman-Enskog derivation of hydrodynamics is given in the framework of Grad's moment equations. Grad's systems are considered as the minimal kinetic models where the Chapman-Enskog method can be studied exactly, thereby providing the basis to compare various approximations in extending the hydrodynamic description beyond the Navier-Stokes approximation. Various techniques, such as the method of partial summation, Padé approximants, and invariance principle are compared both in linear and nonlinear situations.

8.1 The "Ultra-Violet Catastrophe" of the Chapman-Enskog Expansion

Most of the interesting expansions in non-equilibrium statistical physics are divergent. This paraphrase of the well known folklore "Dorfman's theorem" conveys the intrinsic problem of many-body systems: A number of systematic (at the first glance) methods has led to

- An excellent but already known on the phenomenological grounds first approximation;
- Already the next correction, not known phenomenologically and hence of interest, does not exist because of divergence.

There are many examples of this situations: Cluster expansion of the exact collision integral for dense gases leads to divergent approximations of transport coefficients, non-convergent long tails of correlation functions in the Green–Kubo formulae etc.

The derivation of the hydrodynamic equations from a microscopic description is the classical problem of physical kinetics. As is well known, the famous Chapman–Enskog method [70] provides an opportunity to compute a solution from the Boltzmann kinetic equation as a formal series in powers of the Knudsen number ϵ. The parameter ϵ reflects the ratio between the mean free path of a particle, and the scale of variations of the hydrodynamic fields (density, mean flux, and temperature). If the Chapman–Enskog expansion is truncated at a certain order, we obtain subsequently: the Euler hydrodynamics (ϵ^0), the Navier–Stokes hydrodynamics (ϵ^1), the Burnett hydrodynamics (ϵ^2), the

Alexander N. Gorban and Iliya V. Karlin: *Invariant Manifolds for Physical and Chemical Kinetics*, Lect. Notes Phys. **660**, 189–246 (2005)
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super-Burnett hydrodynamics (ϵ^3), etc. The post-Navier–Stokes terms extend the hydrodynamic description beyond the strictly hydrodynamic limit $\epsilon \ll 1$.

However, as it has been first demonstrated by Bobylev [72], even in the simplest regime (one-dimensional linear deviations around the global equilibrium), the Burnett hydrodynamic equations violate the basic physics behind the Boltzmann equation. Namely, sufficiently short acoustic waves are amplified with time instead of decaying. This contradicts the H-theorem, since all near-equilibrium perturbations must decay. The situation does not improve in the next, super-Burnett approximation.

This "ultra-violet catastrophe" which occurs in the lower-order truncations of the Chapman–Enskog expansion creates therefore very serious difficulties in the problem of an extension of the hydrodynamic description into a highly non-equilibrium domain (see [112] for a discussion of other difficulties of the post-Navier–Stokes terms of the Chapman–Enskog expansion). The Euler and the Navier–Stokes approximations remain basic in the hydrodynamic description, while the problem of their extension is one of the central open problems of kinetic theory. The study of approximate solutions based on the Chapman–Enskog method still continues [74].

All this begs for a question: *What is wrong with the Chapman–Enskog method?* At first glance, the failure of the Burnett and of the super-Burnett hydrodynamics may be accounted in favor of a frequently used argument about the asymptotic character of the Chapman–Enskog expansion. However, it is worthwhile to notice here that divergences in the low-order terms of formal expansions are not too surprising. In many occasions, in particular, in quantum field theory [198] and in statistical physics [199], the situation is often improved if one takes into account the very remote terms of the corresponding expansions. Thus, a more constructive viewpoint on the Chapman–Enskog expansion could be to proceed along these lines, and to try to *sum up* the Chapman–Enskog series, at least formally and approximately.

An attempt of this kind of working with the Chapman–Enskog expansion is undertaken in this chapter. The formalities are known to be rather awkward for the Boltzmann equation, and untill now, exact summations of the Chapman–Enskog expansion are known in a very limited number of cases [202]. In this chapter, we shall concentrate on the Chapman–Enskog method as applied to the well known Grad moment equations [201].

The use of the Grad equations for our purpose brings, of course, considerable technical simplifications as compared to the case of the Boltzmann equation but it does not make the problem trivial. Indeed, the Chapman–Enskog method amounts to a nonlinear recurrence procedure even when applied to the simplest, linearized Grad equations. Moreover, as we shall see soon, the Chapman–Enskog expansion for moment systems inherits Bobylev's instability in the low-order approximations. Still, the advantage of our approach is that many explicit results can be obtained and analyzed. In order to

summarize, in this chapter we consider Grad's moment equations as finitely-coupled kinetic models where the problem of reduced description is meaningful, rather than as models of extended hydrodynamics. The latter viewpoint is well known as a microscopic background of the extended irreversible thermodynamics [236, 252].

The outline of this chapter is as follows: after an introduction of the Chapman–Enskog procedure for the linearized Grad equations (Subsect. 8.2), we shall start the discussion with two examples (the linearized one- and three-dimensional 10 moment Grad equations) where the Chapman–Enskog series is summed up exactly in closed form (Sects. 8.3.1 and 8.3.2). These results makes it possible to discuss the features of the Chapman–Enskog solution in the short-wave domain in the framework of the model, and will serve the purpose of testing various approximate methods thereafter. We shall see, in particular, that the "smallness" of the Knudsen number ϵ used to develop the Chapman–Enskog method has no direct meaning in the exact result. Also, it will become clear that finite-order truncations, even provided they are stable, give less opportunities to approximate the solution in a whole, and especially in the short-wave domain.

The exact solutions are, of course, the lucky exceptions, and even for the Grad moment equations the complexity of the Chapman–Enskog method increases rapidly with an increase of the number of the moments taken into account. Further (Sect. 8.4.1) we shall review a technique of summing the Chapman–Enskog expansion *partially*. This technique is heuristic (as are the methods of partial summing in general), but it still removes the Bobylev instability, as well as it qualitatively reproduces the features of the exact solutions in the short-wave limit.

The approach of working in the sections mentioned so far falls into the paradigm of the Taylor-like expansions into powers of the Knudsen number. This viewpoint on the problem of the derivation of the hydrodynamics will be *altered* beginning with Sect. 8.4.2. There we demonstrate that a condition of a *dynamic invariance* which can be realized directly and with no restrictions of the Knudsen number brings us to the same result as the exact summation of the Chapman–Enskog expansion. The Chapman–Enskog method thereafter can be regarded as *one* possibility to solve the resulting invariance equations. Further, we demonstrate that iterative methods provide a reasonable alternative to the Taylor expansion in this problem. Namely, we show that the Newton method has certain advantages over the Chapman–Enskog method (Sect. 8.4.3). We also establish a relationship between the method of partial summation and the Newton method.

The material of further sections serves for an illustrative introduction how the pair "invariance equation + Newton method" can be applied to problems of kinetic theory. The remaining sections of this chapter are devoted to further examples of this approach on the level of the Grad equations. In Sects. 8.4.4 and 8.4.5 we derive and discuss the invariance equations for the linearized

thirteen-moment Grad equations. Section 8.4.6 is devoted to kinetic equations of the Grad type, arising in problems of phonon transport in massive solids at low temperatures. In particular, we demonstrate that the onset of the second sound regime of phonon propagation corresponds to a branching point of the exact sum of the relevant Chapman–Enskog expansion.

In Sect. 8.4.7 we apply the invariance principle to nonlinear Grad equations. We sum up exactly a *subseries* of the Chapman–Enskog expansion, namely, the dominant contribution in the limit of high average velocities. This type of contribution is therefore important for an extension of the hydrodynamic description into the domain of strong shock waves. We present a relevant analysis of the corresponding invariance equation, and, in particular, discuss the nature of singular points of this equation. A brief discussion concludes this chapter. Some of the results presented below were published earlier in [17, 40, 41, 43–45, 205, 237], and summarized in [42].

8.2 The Chapman–Enskog Method for Linearized Grad's Equations

In this section, for the sake of completeness, we introduce linearized Grad's equations and the Chapman–Enskog method for them in the form that will used in the rest of this chapter. Since the Chapman–Enskog method is extensively discussed in a number of books, especially, in the classical monograph [70], our presentation will be brief.

The notation will follow that of the papers [43, 72]. We denote ρ_0, T_0 and $\boldsymbol{u} = 0$ the fixed equilibrium values of density, temperature and averaged velocity (in the appropriate Galilean reference frame), while $\delta\rho$, δT and $\delta\boldsymbol{u}$ are small deviations of the hydrodynamic quantities from their equilibrium values. Grad's moment equations [201] which will appear below, contain the temperature-dependent viscosity coefficient, $\mu(T)$. It is convenient to write $\mu(T) = \eta(T)T$. The functional form of $\eta(T)$ is dictated by the choice of the model for particle interaction. In particular, we have $\eta = \text{const}$ for Maxwell's molecules, and $\eta \sim \sqrt{T}$ for hard spheres.

We use the system of units in which Boltzmann's constant k_B and the particle mass m are equal to one. Let us introduce the following system of dimensionless variables:

$$\begin{aligned} \boldsymbol{u} &= \frac{\delta\boldsymbol{u}}{\sqrt{T_0}}\,, \quad \rho = \frac{\delta\rho}{\rho_0}\,, \quad T = \frac{\delta T}{T_0}\,, \\ \boldsymbol{x} &= \frac{\rho_0}{\eta(T_0)\sqrt{T_0}}\boldsymbol{x}'\,, \quad t = \frac{\rho_0}{\eta(T_0)}t'\,, \end{aligned} \tag{8.1}$$

where $\boldsymbol{x}'$ are spatial coordinates, and t' is time. Three-dimensional thirteen moment Grad's equations, linearized near the equilibrium, take the following form when written in terms of the dimensionless variables (8.1):

$$\begin{aligned}
\partial_t \rho &= -\nabla \cdot \boldsymbol{u} \,, \qquad (8.2)\\
\partial_t \boldsymbol{u} &= -\nabla \rho - \nabla T - \nabla \cdot \boldsymbol{\sigma} \,,\\
\partial_t T &= -\frac{2}{3}(\nabla \cdot \boldsymbol{u} + \nabla \cdot \boldsymbol{q}) \,,\\
\partial_t \boldsymbol{\sigma} &= -\overline{\nabla \boldsymbol{u}} - \frac{2}{5}\overline{\nabla \boldsymbol{q}} - \boldsymbol{\sigma} \,, \qquad (8.3)\\
\partial_t \boldsymbol{q} &= -\frac{5}{2}\nabla T - \nabla \cdot \boldsymbol{\sigma} - \frac{2}{3}\boldsymbol{q} \,.
\end{aligned}$$

In these equations, $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ and $\boldsymbol{q}(\boldsymbol{x}, t)$ are dimensionless quantities corresponding to the stress tensor and to the heat flux, respectively. Further, the gradient ∇ stands for the vector of spatial derivatives $\partial/\partial \boldsymbol{x}$. The dot denotes the standard scalar product, while the overline stands for a symmetric traceless dyad. In particular,

$$\overline{\nabla \boldsymbol{u}} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T - \frac{2}{3} I \nabla \cdot \boldsymbol{u} \,,$$

where I is unit matrix.

Grad's equations (8.2) and (8.3) is the simplest model of a coupling of the hydrodynamic variables, $\rho(\boldsymbol{x}, t)$, $T(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x}, t)$, to the non-hydrodynamic variables $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ and $\boldsymbol{q}(\boldsymbol{x}, t)$. The problem of reduced description is to close the first three equations (8.2), and to get an autonomous system for the hydrodynamic variables alone. In other words, the non-hydrodynamic variables $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ and $\boldsymbol{q}(\boldsymbol{x}, t)$ should be expressed in terms of $\rho(\boldsymbol{x}, t)$, $T(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x}, t)$. The Chapman–Enskog method, as applied for this purpose to Grad's system (8.2) and (8.3), involves the following steps:

First, we introduce a formal parameter ϵ, and write instead of equations (8.3):

$$\begin{aligned}
\partial_t \boldsymbol{\sigma} &= -\overline{\nabla \boldsymbol{u}} - \frac{2}{5}\overline{\nabla \boldsymbol{q}} - \frac{1}{\epsilon}\boldsymbol{\sigma} \,, \qquad (8.4)\\
\partial_t \boldsymbol{q} &= -\frac{5}{2}\nabla T - \nabla \cdot \boldsymbol{\sigma} - \frac{2}{3\epsilon}\boldsymbol{q} \,.
\end{aligned}$$

Second, the Chapman–Enskog solution is found as a formal expansions of the stress tensor and of the heat flux vector:

$$\begin{aligned}
\boldsymbol{\sigma} &= \sum_{n=0}^{\infty} \epsilon^{n+1} \boldsymbol{\sigma}^{(n)} \,; \qquad (8.5)\\
\boldsymbol{q} &= \sum_{n=0}^{\infty} \epsilon^{n+1} \boldsymbol{q}^{(n)} \,.
\end{aligned}$$

The zero-order coefficients, $\boldsymbol{\sigma}^{(0)}$ and $\boldsymbol{q}^{(0)}$, are:

$$\boldsymbol{\sigma}^{(0)} = -\overline{\nabla \boldsymbol{u}} \,, \quad \boldsymbol{q}^{(0)} = -\frac{15}{4}\nabla T \,. \qquad (8.6)$$

Coefficients of order $n \geq 1$ are found from the recurrence procedure:

$$\boldsymbol{\sigma}^{(n)} = -\left\{ \sum_{m=0}^{n-1} \partial_t^{(m)} \boldsymbol{\sigma}^{(n-1-m)} + \frac{2}{5} \overline{\nabla \boldsymbol{q}^{(n-1)}} \right\} , \tag{8.7}$$

$$\boldsymbol{q}^{(n)} = -\frac{3}{2} \left\{ \sum_{m=0}^{n-1} \partial_t^{(m)} \boldsymbol{q}^{(n-1-m)} + \nabla \cdot \boldsymbol{\sigma}^{(n-1)} \right\} ,$$

where $\partial_t^{(m)}$ are recurrently defined *Chapman–Enskog operators*. They act on functions $\rho(\boldsymbol{x}, t)$, $T(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x}, t)$, and on their spatial derivatives, according to the following rule:

$$\partial_t^{(m)} D\rho = \begin{cases} -D\nabla \cdot \boldsymbol{u} & m = 0 \\ 0 & m \geq 1 \end{cases} ; \tag{8.8}$$

$$\partial_t^{(m)} DT = \begin{cases} -\frac{2}{3} D\nabla \cdot \boldsymbol{u} & m = 0 \\ -\frac{2}{3} D\nabla \cdot \boldsymbol{q}^{(m-1)} & m \geq 1 \end{cases} ;$$

$$\partial_t^{(m)} D\boldsymbol{u} = \begin{cases} -D\nabla(\rho + T) & m = 0 \\ -D\nabla \cdot \boldsymbol{\sigma}^{(m-1)} & m \geq 1 \end{cases} .$$

Here D is an arbitrary differential operator with constant coefficients.

Given the initial condition (8.6), the Chapman–Enskog equations (8.7) and (8.8) are recurrently solvable. Finally, by terminating the computation at the order $N \geq 0$, we obtain the Nth order approximations to the expansions (8.5), $\boldsymbol{\sigma}_N$ and $\boldsymbol{q}_N$:

$$\boldsymbol{\sigma}_N = \sum_{n=0}^{N} \epsilon^{n+1} \boldsymbol{\sigma}^{(n)} , \quad \boldsymbol{q}_N = \sum_{n=0}^{N} \epsilon^{n+1} \boldsymbol{q}^{(n)} . \tag{8.9}$$

Substituting these expressions instead of the functions $\boldsymbol{\sigma}$ and $\boldsymbol{q}$ in (8.2), we close the latter to give the hydrodynamic equations of the order N. In particular, $N = 0$ results in the Navier–Stokes approximation, $N = 1$ and $N = 2$ give the Burnett and the super-Burnett approximations, respectively, and so on.

Though the "microscopic" features of Grad's moment equations are, of course, much simpler in comparison to the Boltzmann equation, the Chapman–Enskog procedure just described is not trivial. Our purpose is to study explicitly the features of the gradient expansions like (8.5) in the highly non-equilibrium domain, and, in particular, to find out to what extend the finite-order truncations (8.9) approximate the solution, and what kind of alternative strategies to find approximations are possible. In the following, when referring to Grad's equations, we use the notation $mDnM$, where m is the spatial dimension of the corresponding fields, and n is the number of these fields. For example, the above system is the $3D13M$ Grad's system.

8.3 Exact Summation of the Chapman–Enskog Expansion

8.3.1 The $1D10M$ Grad Equations

In this section, we start the discussion with the exact summation of the Chapman–Enskog series for the simplest Grad's system, the one-dimensional linearized ten-moment equations. Throughout the section we use the hydrodynamic variables $p(x,t) = \rho(x,t) + T(x,t)$ and $u(x,t)$, representing the dimensionless deviations of the pressure and of the average velocity from their equilibrium values (see (8.1)). The starting point is the linearized Grad's equations for p, u, and σ, where σ is the dimensionless xx-component of the stress tensor:

$$\begin{aligned} \partial_t p &= -\frac{5}{3}\partial_x u \,, \\ \partial_t u &= -\partial_x p - \partial_x \sigma \,, \\ \partial_t \sigma &= -\frac{4}{3}\partial_x u - \frac{1}{\epsilon}\sigma \,. \end{aligned} \tag{8.10}$$

The system of equations for three functions is derived from the ten-moment Grad's system (see (8.38) below). Equations (8.10) provides the simplest model of a coupling of the hydrodynamic variables, u and p, to the single non-hydrodynamic variable σ, and corresponds to a heat non-conductive case.

Our goal here is to reduce the description, and to get a closed set of equations with respect to variables p and u only. That is, we have to express the function σ in the terms of spatial derivatives of p and u. The Chapman–Enskog method, as applied to (8.10) results in the following series representation:

$$\sigma = \sum_{n=0}^{\infty} \epsilon^{n+1} \sigma^{(n)} \,. \tag{8.11}$$

The coefficients $\sigma^{(n)}$ are obtained from the following recurrence procedure [43]:

$$\sigma^{(n)} = -\sum_{m=0}^{n-1} \partial_t^{(m)} \sigma^{(n-1-m)} \,, \tag{8.12}$$

where the Chapman–Enskog operators $\partial_t^{(m)}$ act on p, u, and their spatial derivatives as follows:

$$\begin{aligned} \partial_t^{(m)} \partial_x^l u &= \begin{cases} -\partial_x^{l+1} p, & m = 0 \\ -\partial_x^{l+1} \sigma^{(m-1)}, & m \geq 1 \end{cases} \,, \\ \partial_t^{(m)} \partial_x^l p &= \begin{cases} -\frac{5}{3}\partial_x^{l+1} u, & m = 0 \\ 0, & m \geq 1 \end{cases} \,. \end{aligned} \tag{8.13}$$

Here $l \geq 0$ is an arbitrary integer, and $\partial_x^0 = 1$. Finally,

$$\sigma^{(0)} = -\frac{4}{3}\partial_x u, \tag{8.14}$$

which leads to the Navier–Stokes approximation of the stress tensor: $\sigma_{\rm NS} = \epsilon\sigma^{(0)}$.

Because of the somewhat involved structure of the recurrence procedure (8.12) and (8.13), the Chapman–Enskog method is a nonlinear operation even in the simplest model (8.10). Moreover, the Bobylev instability is again present.

Indeed, computing the coefficients $\sigma^{(1)}$ and $\sigma^{(2)}$ on the basis of (8.12), we obtain:

$$\sigma_{\rm B} = \epsilon\sigma^{(0)} + \epsilon^2\sigma^{(1)} = -\frac{4}{3}\left(\epsilon\partial_x u + \epsilon^2\partial_x^2 p\right) , \tag{8.15}$$

and

$$\sigma_{\rm SB} = \epsilon\sigma^{(0)} + \epsilon^2\sigma^{(1)} + \epsilon^3\sigma^{(2)} = -\frac{4}{3}\left(\epsilon\partial_x u + \epsilon^2\partial_x^2 p + \frac{1}{3}\epsilon^3\partial_x^3 u\right) , \tag{8.16}$$

for the Burnett and the super-Burnett approximations, respectively. Now we can substitute each of the approximations, $\sigma_{\rm NS}$, $\sigma_{\rm B}$, and $\sigma_{\rm SB}$ for σ in the second equation of the set (8.10). The equations thus obtained, together with the equation for density ρ, form the closed systems of the hydrodynamic equations of the Navier–Stokes, Burnett, and super-Burnett levels. To see the properties of the resulting equations, we compute the dispersion relation for the hydrodynamic modes. Using a new space-time scale, $x' = \epsilon^{-1}x$, and $t' = \epsilon^{-1}t$, and representing $u = u_k\varphi(x', t')$, and $p = p_k\varphi(x', t')$, where $\varphi(x', t') = \exp(\omega t' + ikx')$, and k is a real-valued wave vector, we obtain the following dispersion relations $\omega(k)$ from the condition of a non-trivial solvability of the corresponding linear system with respect to u_k and p_k:

$$\omega_\pm = -\frac{2}{3}k^2 \pm \frac{1}{3}i|k|\sqrt{4k^2 - 15} , \tag{8.17}$$

for the Navier–Stokes approximation,

$$\omega_\pm = -\frac{2}{3}k^2 \pm \frac{1}{3}i|k|\sqrt{8k^2 + 15} , \tag{8.18}$$

for the Burnett approximation (8.15), and

$$\omega_\pm = \frac{2}{9}k^2(k^2 - 3) \pm \frac{1}{9}i|k|\sqrt{4k^6 - 24k^4 - 72k^2 - 135} , \tag{8.19}$$

for the super-Burnett approximation (8.16).

These examples demonstrate that the real part $\mathrm{Re}(\omega_\pm(k)) \le 0$ for the Navier–Stokes (8.17) and for the Burnett (8.18) approximations, for all wave vectors. Thus, these approximations describe attenuating acoustic waves. However, for the super-Burnett approximation, the function $\mathrm{Re}(\omega_\pm(k))$ (8.19)

becomes positive as soon as $|k| > \sqrt{3}$. That is, the equilibrium point is stable within the Navier–Stokes and the Burnett approximation, and it becomes unstable within the super-Burnett approximation for sufficiently short waves. Similar to the case of the Bobylev instability of the Burnett hydrodynamics for the Boltzmann equation, the latter result contradicts the dissipative properties of the Grad system (8.10): the spectrum of the full $1D10M$ system (8.10) is stable for arbitrary k.

Our goal now is to sum up the series (8.11) in closed form. Firstly, we should make some preparations.

As demonstrated in [43] (see also below), the functions $\sigma^{(n)}$ in (8.11) and (8.12) have the following explicit structure to arbitrary order $n \geq 0$:

$$\begin{aligned} \sigma^{(2n)} &= a_n \partial_x^{2n+1} u \,, \\ \sigma^{(2n+1)} &= b_n \partial_x^{2(n+1)} p \,, \end{aligned} \tag{8.20}$$

where the coefficients a_n and b_n are determined through the recurrence procedure (8.12), and (8.13). The Chapman–Enskog procedure (8.12) and (8.13) can be represented in terms of the real-valued coefficients a_n and b_n (8.20).

Knowing the structure (8.20) of the coefficients of the Chapman–Enskog expansion (8.11), we can write down its formal sum. It is convenient to use the Fourier variables introduced above which amounts essentially to the change $\epsilon\partial_x \to ik$. Substituting expression (8.20) into the Chapman–Enskog series (8.11), we obtain the following formal expression for the Fourier image of the sum:

$$\sigma_k = ikA(k^2)u_k - k^2 B(k^2) p_k \,, \tag{8.21}$$

where the functions $A(k^2)$ and $B(k^2)$ are formal power series with the coefficients (8.20):

$$\begin{aligned} A(k^2) &= \sum_{n=0}^{\infty} a_n (-k^2)^n \,, \\ B(k^2) &= \sum_{n=0}^{\infty} b_n (-k^2)^n \,. \end{aligned} \tag{8.22}$$

Thus, the question of the summation of the Chapman–Enskog series (8.11) amounts to finding the two functions, $A(k^2)$ and $B(k^2)$ (8.22). Knowing them, the dispersion relation for the hydrodynamic modes can be derived:

$$\omega_{\pm} = \frac{k^2 A}{2} \pm \frac{|k|}{2} \sqrt{k^2 A^2 - \frac{20}{3}(1 - k^2 B)} \,. \tag{8.23}$$

We shall concentrate now on the problem of deriving $A(k^2)$ and $B(k^2)$ (8.22) in closed form. For this purpose, we shall first express the Chapman–Enskog procedure (8.12) and (8.13) in terms of the coefficients a_n and b_n (8.20). At the same time, our derivation will constitute proof for the structure (8.20).

It is convenient to start with the Fourier representation of (8.12) and (8.13). Writing $u = u_k \exp(ikx)$, $p = p_k \exp(ikx)$, and $\sigma = \sigma_k \exp(ikx)$, we obtain:

$$\partial_t^{(m)} u_k = \begin{cases} -ikp_k, & m = 0 \\ -ik\sigma_k^{(m-1)}, & m \geq 1 \end{cases} , \tag{8.24}$$
$$\partial_t^{(m)} p_k = \begin{cases} -\frac{5}{3} iku_k, & m = 0 \\ 0, & m \geq 1 \end{cases} ,$$

while

$$\sigma_k^{(n)} = -\sum_{m=0}^{n-1} \partial_t^{(m)} \sigma_k^{(n-1-m)} , \tag{8.25}$$

and

$$\begin{aligned} \sigma_k^{(2n)} &= a_n (-k^2)^n iku_k , \\ \sigma_k^{(2n+1)} &= b_n (-k^2)^n (-k^2) p_k . \end{aligned} \tag{8.26}$$

The Navier–Stokes and the Burnett approximations give $a_0 = -\frac{4}{3}$, and $b_0 = -\frac{4}{3}$. Thus, the structure (8.26) is proved for $n = 0$.

The further derivation relies on induction. Let us assume that the ansatz (8.26) is proven up to the order n. Computing the coefficient $\sigma_k^{(2(n+1))}$ from (8.25), we have:

$$\sigma_k^{(2(n+1))} = -\partial_t^{(0)} \sigma_k^{(2n+1)} - \sum_{m=0}^{n} \partial_t^{(2m+1)} \sigma_k^{(2(n-m))} - \sum_{m=1}^{n} \partial_t^{(2m)} \sigma_k^{(2(n-m)+1)} . \tag{8.27}$$

Due to the assumption of the induction, we can adopt the form of the coefficients $\sigma_k^{(j)}$ (8.26) in all the terms on the right hand side of (8.27). On the basis of (8.26) and (8.24), we conclude that each term in the last sum of (8.27) is equal to zero. Further, the term $\partial_t^{(0)} \sigma_k^{(2n+1)}$ gives the linear contribution:

$$\partial_t^{(0)} \sigma_k^{(2n+1)} = \partial_t^{(0)} b_n (-k^2)^n (-k^2) p_k = -\frac{5}{3} b_n (-k^2)^{n+1} iku_k ,$$

while the terms in the remaining sum contribute nonlinearly:

$$\partial_t^{(2m+1)} \sigma_k^{(2(n-m))} = a_{n-m} (-k^2)^{n-m} ik \partial_t^{(2m+1)} u_k = -a_{n-m} a_m (-k^2)^{n+1} iku_k .$$

Substituting the last two expressions into (8.27), we see that it has just the same structure as the coefficient $\sigma_k^{(2(n+1))}$ in (8.26). Thus, we obtain the first recurrence equation:

$$a_{n+1} = \frac{5}{3} b_n + \sum_{m=0}^{n} a_{n-m} a_m .$$

Computing the coefficient $\sigma_k^{(2(n+1)+1)}$ by the same pattern, we come to the second recurrence equation, and the Chapman–Enskog procedure (8.12) and (8.13) can be reformulated in terms of the coefficients a_n and b_n (8.20):

$$a_{n+1} = \frac{5}{3} b_n + \sum_{m=0}^{n} a_{n-m} a_m \,, \tag{8.28}$$
$$b_{n+1} = a_{n+1} + \sum_{m=0}^{n} a_{n-m} b_m \,.$$

The initial condition for this set of equations is dictated by the Navier–Stokes and the Burnett terms:

$$a_0 = -\frac{4}{3} \,, \quad b_0 = -\frac{4}{3} \tag{8.29}$$

Our goal now is to compute the functions A and B (8.22) on the basis of the recurrence equations (8.28). At this point, it is worthwhile to notice that the usual way of dealing with the recurrence system (8.28) would be either to truncate it at a certain n, or to calculate all the coefficients explicitly, and substitute the result into the power series (8.22). Both approaches are not successful here. Indeed, retaining the coefficients a_0, b_0, and a_1 gives the super-Burnett approximation (8.16) which has the Bobylev short-wave instability, and there is no guarantee that the same failure will not occur in the higher-order truncation. On the other hand, a term-by-term computation of the whole set of coefficients a_n and b_n is a nontrivial task due to the nonlinearity in (8.28).

Fortunately, another route is possible. Multiplying both the equations in (8.28) with $(-k^2)^{n+1}$, and performing a formal summation in n from zero to infinity, we arrive at the following expressions:

$$A - a_0 = -k^2 \left\{ \frac{5}{3} B + \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n-m} (-k^2)^{n-m} a_m (-k^2)^m \right\} \,, \tag{8.30}$$
$$B - b_0 = A - a_0 - k^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n-m} (-k^2)^{n-m} b_m (-k^2)^m \,.$$

Now we notice that

$$\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{n} a_{n-m} (-k^2)^{n-m} a_m (-k^2)^m = A^2 \,, \tag{8.31}$$
$$\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{n} a_{n-m} (-k^2)^{n-m} b_m (-k^2)^m = AB \,.$$

Taking into account the initial condition (8.29), equation (8.30) yields a pair of coupled quadratic equations for the functions A and B:

$$A = -\frac{4}{3} - k^2 \left(\frac{5}{3} B + A^2 \right) , \tag{8.32}$$
$$B = A(1 - k^2 B) .$$

The result (8.32) concludes essentially the question of the computation of functions A and B (8.22). Still, further simplifications are possible. In particular, it is convenient to reduce the consideration to a single function. Solving system (8.32) for B, and introducing a new function, $X(k^2) = k^2 B(k^2)$, we obtain an equivalent cubic equation:

$$-\frac{5}{3}(X-1)^2 \left(X + \frac{4}{5} \right) = \frac{X}{k^2} . \tag{8.33}$$

Since A and B (8.22) are real-valued, we are only interested in the real-valued roots of (8.33).

An elementary analysis of this equation brings the following result: *the real-valued root $X(k^2)$ of (8.33) is unique and negative for all finite values k^2*. Moreover, the function $X(k^2)$ is a monotonic function of k^2 (Fig. 8.1). The limiting values are:

$$\lim_{|k|\to 0} X(k^2) = 0 , \qquad \lim_{|k|\to\infty} X(k^2) = -0.8 . \tag{8.34}$$

Under the conditions just mentioned, the function under the root in (8.23) is negative for all values of the wave vector k, including the limits, and we come to the following dispersion law:

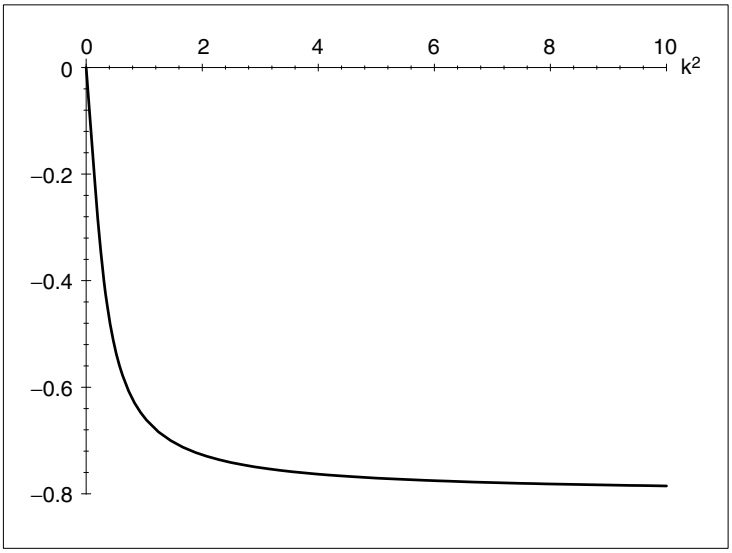

Fig. 8.1. Real-valued root of (8.33) as a function of k^2

$$\omega_{\pm} = \frac{X}{2(1-X)} \pm i\frac{|k|}{2}\sqrt{\frac{5X^2 - 16X + 20}{3}}\,, \tag{8.35}$$

where $X = X(k^2)$ is the real-valued root of equation (8.33). Since $X(k^2)$ is negative for all $|k| > 0$, the attenuation rate, $\mathrm{Re}(\omega_{\pm})$, is negative for all $|k| > 0$, and the exact acoustic spectrum of the Chapman–Enskog procedure *is stable for arbitrary wave lengths*. In the short-wave limit, from (8.35) we obtain:

$$\lim_{|k|\to\infty} \omega_{\pm} = -\frac{2}{9} \pm i|k|\sqrt{3}\,. \tag{8.36}$$

The characteristic equation of the original Grad equations (8.10) reads:

$$3\omega^3 + 3\omega^2 + 9k^2\omega + 5k^2 = 0\,. \tag{8.37}$$

The two complex-conjugate roots of this equation correspond to the hydrodynamic modes, while for the non-hydrodynamic real mode, $\omega_{nh}(k)$, $\omega_{nh}(0) = -1$, and $\omega_{nh} \to -0.5$ as $|k| \to \infty$. Recall that the non-hydrodynamic modes of the Grad equations are characterized by the common property that for them $\omega(0) \neq 0$. These modes are irrelevant to the Chapman–Enskog method. As the final comment here, (8.36) demonstrates that the exact attenuation rate, $\mathrm{Re}(\omega_{\pm})$, tends to a finite value, $-\frac{2}{9} \approx -0.22$ as $|k| \to \infty$. This asymptotic behavior is in a complete agreement with the data for the hydrodynamic branch of the spectrum (8.37) of the original Grad equations (8.10). The attenuation rates (real parts of the dispersion relations $\omega_{\pm}$ for the Burnett (8.18), the super-Burnett (8.19), the exact Chapman–Enskog solution (8.35), are compared to each other in Fig. 8.2. In this figure, we also represent the attenuation rates of the hydrodynamic and non-hydrodynamic mode of the Grad equations (8.37). The results of this section lead to the following conclusion:

(i) The proposed approach provides a way to deal with the problem of *summation* of the Chapman–Enskog expansion. The exact dispersion relation (8.35) of the Chapman–Enskog procedure is demonstrated to be stable for all wave lengths, while the Bobylev instability is present on the level of the super-Burnett approximation. Moreover, it can be demonstrated that the function X (the real root of (8.33)) is a real-valued analytic function of k. Thus, the treatment of the formal expansions performed above is justified.

(ii) The exact result of the Chapman–Enskog procedure has a clear non-polynomial character. Indeed, this follows directly from (8.34): the function $X(k^2)$ cannot be a polynomial because it maps the axis k into a segment $[0, -0.8]$. As a conjecture here, the resulting exact hydrodynamics is *essentially* nonlocal in space. For this reason, even if the hydrodynamic equations of a certain level of the approximation *is* stable, it cannot reproduce the non-polynomial behavior for sufficiently short waves.

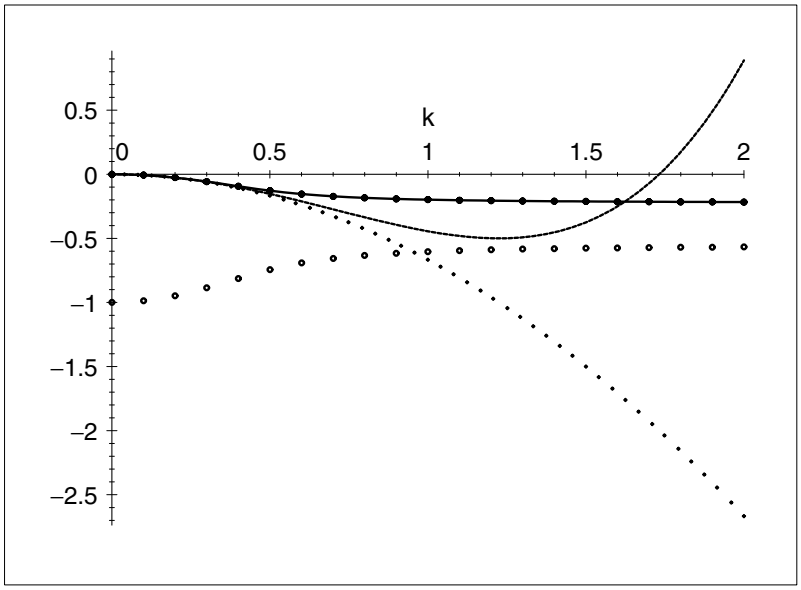

Fig. 8.2. Attenuation rates for the $1D10M$ Grad system. *Solid*: Exact summation of the Chapman–Enskog Expansion. *Dots*: The Navier–Stokes approximation. *Dash*: The super–Burnett approximation. *Circles*: Hydrodynamic and non-hydrodynamic modes of the $1D10M$ Grad system

(iii) The result of this section demonstrates that, at least in some cases, the sum of the Chapman–Enskog series amounts to a quite regular function, and the "smallness" of the Knudsen number ϵ used to develop the Chapman–Enskog procedure (8.12) *is no longer necessary*.

8.3.2 The $3D10M$ Grad Equations

In this section we generalize our considerations of the Chapman–Enskog method to the three-dimensional linearized 10-moment Grad equations [201]. The Chapman–Enskog series for the stress tensor, which is again due to a nonlinear procedure, will be summed up in closed form. The method used follows essentially the one discussed above, though the computations are slightly more extensive. The reason to consider this example is that we would like to know what happens to the diffusive hydrodynamic mode in the short-wave domain.

Throughout this section, we use the variables (8.1), and p and $\boldsymbol{u}$ are dimensionless deviations of pressure and of mean flux from their equilibrium values, respectively. The point of departure is the set of the three-dimensional linearized Grad equations for the p, $\boldsymbol{u}$, and $\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is a dimensionless stress tensor:

$$\begin{aligned} \partial_t p &= -\frac{5}{3}\nabla\cdot\boldsymbol{u}\ , \\ \partial_t \boldsymbol{u} &= -\nabla p - \nabla\cdot\boldsymbol{\sigma}\ , \\ \partial_t \boldsymbol{\sigma} &= -\overline{\nabla\boldsymbol{u}} - \frac{1}{\epsilon}\boldsymbol{\sigma}\ . \end{aligned} \tag{8.38}$$

Equation (8.38) provides a simple model of a coupling of the hydrodynamic variables, $\boldsymbol{u}$ and p, to the non-hydrodynamic variable $\boldsymbol{\sigma}$. These equations are suitable for an application of the Chapman–Enskog procedure. Therefore, our goal here is not to investigate the properties of (8.38) as they are, but to reduce the description, and to get a closed set of equations with respect to the variables p and $\boldsymbol{u}$ only. That is, we have to express $\boldsymbol{\sigma}$ in terms of spatial derivatives of p and of $\boldsymbol{u}$. The Chapman–Enskog method, as applied to (8.38) results in the following:

$$\boldsymbol{\sigma} = \sum_{n=0}^{\infty} \epsilon^{n+1}\boldsymbol{\sigma}^{(n)}\ . \tag{8.39}$$

The coefficients $\boldsymbol{\sigma}^{(n)}$ are due to the following recurrence procedure:

$$\boldsymbol{\sigma}^{(n)} = -\sum_{m=0}^{n-1} \partial_t^{(m)}\boldsymbol{\sigma}^{(n-1-m)}\ , \tag{8.40}$$

where the Chapman–Enskog operators $\partial_t^{(m)}$ act on the functions p and $\boldsymbol{u}$, and on their derivatives, as follows:

$$\begin{aligned} \partial_t^{(m)} D\boldsymbol{u} &= \begin{cases} -D\nabla p, & m = 0 \\ -D\nabla\cdot\boldsymbol{\sigma}^{(m-1)}, & m \geq 1 \end{cases}\ , \\ \partial_t^{(m)} Dp &= \begin{cases} -\frac{5}{3}D\nabla\cdot\boldsymbol{u}, & m = 0 \\ 0, & m \geq 1 \end{cases}\ . \end{aligned} \tag{8.41}$$

Here D is an arbitrary differential operator $D = \prod_{i=1}^{3}\partial_i^{l_i}$, while l_i is an arbitrary integer, and $\partial_i^0 = 1$. Finally, $\boldsymbol{\sigma}^{(0)} = -\overline{\nabla\boldsymbol{u}}$, which leads to the Navier–Stokes approximation.

Our goal is to sum up the series (8.39) in closed form.

The terms $\boldsymbol{\sigma}^{(n)}$ in equations (8.39), (8.40), and (8.41), have the following explicit structure for arbitrary order $n \geq 0$ (a generalization of (8.20) to the three-dimensional case):

$$\begin{aligned} \boldsymbol{\sigma}^{(2n)} &= a_n \Delta^n \overline{\nabla\boldsymbol{u}} + b_n \Delta^{n-1} G\nabla\cdot\boldsymbol{u}\ , \\ \boldsymbol{\sigma}^{(2n+1)} &= c_n \Delta^n G p\ , \end{aligned} \tag{8.42}$$

where $\Delta = \nabla\cdot\nabla$ is the Laplace operator, and the operator G has the form:

$$G = \nabla\nabla - \frac{1}{3}I\Delta = \frac{1}{2}\overline{\nabla\nabla}\ . \tag{8.43}$$

The real-valued and yet unknown coefficients a_n, b_n, and c_n in (8.42) are due to the recurrence procedure (8.40), and (8.41). Knowing the structure of the coefficients of the Chapman–Enskog series (8.42), we can reformulate the Chapman–Enskog solution in terms of a self-consistent recurrence procedure for the coefficients a_n, b_n, and c_n. Let us consider this derivation in more detail.

The point of departure is the Fourier representation of the recurrence equations (8.40), (8.41), and (8.42). Writing

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}_k \exp(i\boldsymbol{k}\cdot\boldsymbol{x}) \,, \\ p &= p_k \exp(i\boldsymbol{k}\cdot\boldsymbol{x}) \,, \\ \boldsymbol{\sigma}^{(n)} &= \boldsymbol{\sigma}_k^{(n)} \exp(i\boldsymbol{k}\cdot\boldsymbol{x}) \,, \end{aligned}$$

and introducing the unit vector $\boldsymbol{e}_k$ directed along $\boldsymbol{k}$ ($\boldsymbol{k} = k\boldsymbol{e}_k$), equations (8.40), (8.41), and (8.42) can be rewritten as:

$$\boldsymbol{\sigma}_k^{(n)} = -\sum_{m=0}^{n-1} \partial_t^{(m)} \boldsymbol{\sigma}_k^{(n-1-m)} \,, \tag{8.44}$$

$$\begin{aligned} \partial_t^{(m)} D_k \boldsymbol{u}_k &= \begin{cases} -D_k i\boldsymbol{k} p_k, & m = 0 \\ -D_k i\boldsymbol{k}\cdot\boldsymbol{\sigma}_k^{(m-1)}, & m \geq 1 \end{cases} \,, \\ \partial_t^{(m)} D_k p_k &= \begin{cases} -\frac{5}{3} D_k i\boldsymbol{k}\cdot\boldsymbol{u}_k, & m = 0 \\ 0, & m \geq 1 \end{cases} \,. \end{aligned} \tag{8.45}$$

where D_k is an arbitrary tensor $D_k = \prod_{s=1}^{3} (ik_s)^{l_s}$, and

$$\begin{aligned} \boldsymbol{\sigma}_k^{(2n)} &= (-k^2)^n (a_n i\overline{\boldsymbol{k}\boldsymbol{u}} + b_n i\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u})) \,, \\ \boldsymbol{\sigma}_k^{(2n+1)} &= c_n (-k^2)^{n+1} \boldsymbol{g}_k p_k \,, \end{aligned} \tag{8.46}$$

where

$$\boldsymbol{g}_k = \left(\boldsymbol{e}_k\boldsymbol{e}_k - \frac{1}{3} I \right) = \frac{1}{2}\overline{\boldsymbol{e}_k\boldsymbol{e}_k} \,. \tag{8.47}$$

From the form of the Navier–Stokes approximation, $\boldsymbol{\sigma}_k^{(0)}$, it follows that $a_0 = -1$ and $b_0 = 0$, while a direct computation of the Burnett approximation leads to:

$$\boldsymbol{\sigma}_k^{(1)} = \frac{1}{2} k^2 \boldsymbol{g}_k p_k \,. \tag{8.48}$$

Thus, we have $c_0 = -\frac{1}{2}$which proves the ansatz (8.42) for $n = 0$ in both the even and the odd orders.

The rest of the proof relies on induction. Let the structure (8.46) be proven up to the order n. The computation of the next, $n+1$ order coefficient $\boldsymbol{\sigma}_k^{(2(n+1))}$, involves only terms of lower order. From (8.44) we obtain:

$$\boldsymbol{\sigma}_k^{(2(n+1))} = -\partial_t^{(0)}\boldsymbol{\sigma}_k^{(2n+1)} - \sum_{m=1}^{2n+1}\partial_t^{(m)}\boldsymbol{\sigma}_k^{(2n+1-m)} \,. \tag{8.49}$$

The first term in the right hand side depends linearly on the coefficients c_n:

$$\begin{aligned} -\partial_t^{(0)}\boldsymbol{\sigma}_k^{(2n+1)} &= -c_n(-k^2)^{n+1}\boldsymbol{g}_k\partial_t^{(0)}p_k \\ &= \frac{5}{3}c_n(-k^2)^{n+1}i\boldsymbol{g}_k\boldsymbol{k}\cdot\boldsymbol{u}_k \,. \end{aligned} \tag{8.50}$$

The remaining terms on the right hand side of (8.49) contribute nonlinearly. Splitting the even and the odd orders of the Chapman–Enskog operators $\partial_t^{(m)}$, we rewrite the sum in (8.49):

$$-\sum_{m=1}^{2n+1}\partial_t^{(m)}\boldsymbol{\sigma}_k^{(2n+1-m)} = -\sum_{l=1}^{n}\partial_t^{(2l)}\boldsymbol{\sigma}_k^{(2(n-l)+1)} - \sum_{l=0}^{n}\partial_t^{(2l+1)}\boldsymbol{\sigma}_k^{(2(n-l))} \,. \tag{8.51}$$

Due to (8.46) and (8.45), each term in the first sum is equal to zero, and we are left only with the second sum:

$$\partial_t^{(2l+1)}\boldsymbol{\sigma}_k^{(2(n-l))} = (-k^2)^{n-l}(a_{n-l}i\overline{\boldsymbol{k}\partial_t^{(2l+1)}\boldsymbol{u}_k} + b_{n-l}i\boldsymbol{g}_k\boldsymbol{k}\cdot\partial_t^{(2l+1)}\boldsymbol{u}_k) \,, \tag{8.52}$$

while

$$\partial_t^{(2l+1)}\boldsymbol{u}_k = -(-k^2)^{l+1}\left(a_l\boldsymbol{u}_k + \frac{1}{3}(a_l+2b_l)\boldsymbol{e}_k(\boldsymbol{e}_k\cdot\boldsymbol{u}_k)\right) \,. \tag{8.53}$$

In the last expression, use of the following identities was made:

$$\begin{aligned} \boldsymbol{k}\cdot\overline{\boldsymbol{k}\boldsymbol{u}_k} &= k^2\left(\boldsymbol{u}_k + \frac{1}{3}\boldsymbol{e}_k(\boldsymbol{e}_k\cdot\boldsymbol{u}_k)\right) \,, \\ \boldsymbol{k}\cdot\boldsymbol{g}_k &= \frac{2}{3}\boldsymbol{k} \,. \end{aligned} \tag{8.54}$$

Substituting (8.53) into the right hand side of (8.52), and thereafter substituting the result into the right hand side of (8.51), we obtain the following in the right hand side of (8.49):

$$\begin{aligned} \boldsymbol{\sigma}_k^{(2(n+1))} &= (-k^2)^{n+1}\left(\sum_{m=0}^{n}a_{n-m}a_m\right)i\overline{\boldsymbol{k}\boldsymbol{u}_k} + (-k^2)^{n+1}\left(\frac{5}{3}c_n\right. \\ &\left. + \sum_{m=0}^{n}\left\{\frac{1}{3}(2a_{n-m}+b_{n-m})(a_m+2b_m) + a_{n-m}b_m\right\}\right)i\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k) \,. \end{aligned} \tag{8.55}$$

The functional structure of the right hand side of this expression is the same as that of the first equation in the set (8.46), and thus we obtain the first recurrence equation:

$$a_{n+1}\overline{\boldsymbol{k}\boldsymbol{u}_k} + b_{n+1}\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k) = \left(\sum_{m=0}^{n} a_{n-m}a_m\right)\overline{\boldsymbol{k}\boldsymbol{u}_k} \tag{8.56}$$
$$+\left(\frac{5}{3}c_n + \sum_{m=0}^{n}\left\{\frac{1}{3}(2a_{n-m}+b_{n-m})(a_m+2b_m) + a_{n-m}b_m\right\}\right)\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)\ .$$

Considering in the same way the coefficient $\boldsymbol{\sigma}_k^{(2(n+1)+1)}$, we come to the second recurrence equation,

$$c_{n+1} = 2a_{n+1} + b_{n+1} + \frac{2}{3}\sum_{m=0}^{n}(2a_{n-m}+b_{n-m})c_m\ . \tag{8.57}$$

Thus, the complete set of the recurrence equations is given by (8.56) and (8.57). Equation (8.56) is equivalent to a pair of scalar equations. Indeed, introducing new variables,

$$\begin{aligned} r_n &= \frac{2}{3}c_n\ , \\ q_n &= \frac{2}{3}(2a_n + b_n)\ , \end{aligned} \tag{8.58}$$

and using the identity,

$$\overline{\boldsymbol{k}\boldsymbol{u}_k} = (\overline{\boldsymbol{k}\boldsymbol{u}_k} - 2\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)) + 2\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)\ ,$$

and also noticing that

$$\boldsymbol{g}_k : (\overline{\boldsymbol{k}\boldsymbol{u}_k} - 2\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)) = 0\ ,$$

where : denotes the double contraction of tensors, we arrive in (8.56) and (8.57) at the following three scalar recurrence relations in terms the coefficients r_n, q_n, and a_n:

$$\begin{aligned} r_{n+1} &= q_{n+1} + \sum_{m=0}^{n} q_{n-m}r_m \\ q_{n+1} &= \frac{5}{3}r_n + \sum_{m=0}^{n} q_{n-m}q_m \\ a_{n+1} &= \sum_{m=0}^{n} a_{n-m}a_m \end{aligned} \tag{8.59}$$

The initial condition for this system is provided by the explicit form of the Navier–Stokes and the Burnett approximations, and reads:

$$r_0 = -4/3, \quad q_0 = -4/3, \quad a_0 = -1\ . \tag{8.60}$$

The recurrence relations (8.59) are completely equivalent to the original Chapman–Enskog procedure (8.40) and (8.41). In the one-dimensional case, the recurrence system (8.59) reduces to the first two equations for r_n and q_n. In this case, the system of recurrence equations is identical (up to the notations) to the recurrence system (8.28), considered in the preceding section. For what follows, it is important to notice that the recurrence equation for the coefficients a_n is decoupled from the equations for the coefficients r_n and q_n.

Now we shall express the Chapman–Enskog series of the stress tensor (8.39) in terms of r_n, q_n, and a_n. Using again the Fourier transform, and substituting (8.42) into the right hand side of (8.39), we derive:

$$\boldsymbol{\sigma}_k = A(k^2)(\overline{\boldsymbol{k}\boldsymbol{u}_k} - 2\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)) + \frac{3}{2}Q(k^2)\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k) - \frac{3}{2}k^2R(k^2)\boldsymbol{g}_k p_k \ , \quad (8.61)$$

From here on, we use a new spatial scale which amounts to $\boldsymbol{k}' = \epsilon\boldsymbol{k}$, and drop the prime. The functions $A(k^2)$, $Q(k^2)$, and $R(k^2)$ in (8.61) are defined by the power series with the coefficients due to (8.59):

$$A(k^2) = \sum_{n=0}^{\infty} a_n(-k^2)^n \ , \quad (8.62)$$

$$Q(k^2) = \sum_{n=0}^{\infty} q_n(-k^2)^n \ ,$$

$$R(k^2) = \sum_{n=0}^{\infty} r_n(-k^2)^n \ .$$

Thus, the question of summation of the Chapman–Enskog series (8.39) amounts to finding the three functions, $A = A(k^2)$, $Q = Q(k^2)$, and $R = R(k^2)$ (8.62) in the three- and two-dimensional cases, or to the two functions, $Q(k^2)$, and $R(k^2)$ in the one-dimensional case.

Now we shall focus on computing the functions (8.62) from the recurrence equations (8.59). At this point, it is worthwhile to notice again that a truncation at a certain n is not successful. Indeed, already in the one-dimensional case, retaining the coefficients q_0, r_0, and q_1 leads to the super-Burnett approximation (8.16) which has the short-wave instability for $k^2 > 3$, as it was demonstrated in the preceding section, and there is no guarantee that the same will not occur in a higher-order truncation.

Fortunately, the approach introduced in the preceding section works again. Multiplying each of the equations in (8.62) with $(-k^2)^{n+1}$, and performing a summation in n from zero to infinity, we derive:

$$Q - q_0 = -k^2\left\{\frac{5}{3}R + \sum_{n=0}^{\infty}\sum_{m=0}^{n} q_{n-m}(-k^2)^{n-m}q_m(-k^2)^m\right\} \ , \quad (8.63)$$

$$R - r_0 = Q - q_0 - k^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} q_{n-m}(-k^2)^{n-m} r_m (-k^2)^m \ ,$$
$$A - a_0 = -k^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m} a_m (-k^2)^m \ .$$

Now we notice that

$$\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m} a_m (-k^2)^m = A^2 \ , \tag{8.64}$$
$$\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{n} q_{n-m}(-k^2)^{n-m} r_m (-k^2)^m = QR \ ,$$
$$\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{n} q_{n-m}(-k^2)^{n-m} q_m (-k^2)^m = Q^2 \ .$$

Taking into account the initial conditions (8.60), and also using (8.64), we derive from (8.63) the following three quadratic equations for the functions A, R, and Q:

$$Q = -\frac{4}{3} - k^2 \left(\frac{5}{3} R + Q^2 \right) \ , \tag{8.65}$$
$$R = Q(1 - k^2 R) \ ,$$
$$A = -(1 + k^2 A^2) \ .$$

The result (8.65) concludes essentially the question of computation of functions (8.62) in closed form. Still, further simplifications are possible. In particular, it is convenient to use a single unknown function, $X(k^2) = k^2 R(k^2)$, in the first two equations in the system (8.65). We again obtain an equivalent cubic equation:

$$-\frac{5}{3}(X-1)^2 (X + \frac{4}{5}) = \frac{X}{k^2} \ , \tag{8.66}$$

which coincides with (8.66) of the previous section. We shall also rewrite the third equation of (8.65) using a function $Y(k^2) = k^2 A(k^2)$:

$$Y(1 + Y) = -k^2 \ . \tag{8.67}$$

The functions in (8.62) can now be straightforwardly expressed in terms of the relevant solutions to (8.66) and (8.67). Since all functions in (8.62) are real-valued functions, we are interested only in the real-valued roots of the algebraic equations (8.66) and (8.67).

The relevant analysis of the cubic equation (8.66) was already performed above: the real-valued root $X(k^2)$ is unique and negative for all finite values of k^2. Limiting values of the function $X(k^2)$ at $k \to 0$ and at $k \to \infty$ are given by (8.34):

$$\lim_{k\to 0} X(k^2) = 0\,, \quad \lim_{k\to\infty} X(k^2) = -\frac{4}{5}\,.$$

The quadratic equation (8.67) has no real-valued solutions for $k^2 > \frac{1}{4}$, and it has two real-valued solution for each k^2, where $k^2 < \frac{1}{4}$. We denote $k_c = \frac{1}{2}$ the corresponding critical value of the wave vector. For $k = 0$, one of these roots is equal to zero, while the other is equal to one. The asymptotics $Y \to 0$, as $k \to 0$, answers the question which of these two roots of (8.67) is relevant to the Chapman–Enskog solution, and we derive:

$$Y = \begin{cases} -\frac{1}{2}\left(1 - \sqrt{1-4k^2}\right) & k < k_c \\ \text{none} & k > k_c \end{cases} \tag{8.68}$$

The function Y (8.68) is negative for $k \le k_c$.

From now on, X and Y will denote the relevant roots of (8.66) and (8.67) just discussed. The Fourier image of the expression $\nabla\cdot\boldsymbol{\sigma}$ follows from (8.61):

$$i\boldsymbol{k}\cdot\boldsymbol{\sigma}_k = Y((\boldsymbol{e}_k\cdot\boldsymbol{u}_k)\boldsymbol{e}_k - \boldsymbol{u}_k) - \frac{X}{1-X}(\boldsymbol{e}_k\cdot\boldsymbol{u}_k)\boldsymbol{e}_k - iX\boldsymbol{k}p_k\,. \tag{8.69}$$

The latter expression contributes to the right-hand side of the second of equations in the Grad system (8.38) (more specifically, it contributes to the corresponding Fourier transform of this equation). Knowing (8.69), we can calculate the dispersion $\omega(\boldsymbol{k})$ of the plane waves $\sim\exp\{\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}\}$ which now follows from the exact solution of the Chapman–Enskog procedure. The calculation of the dispersion relation amounts to an evaluation of the determinant of a $(d+1)\times(d+1)$ matrix, and is quite standard (see, e.g. [240]). We therefore provide only the final result. The exact dispersion relation of the hydrodynamic modes reads:

$$(\omega - Y)^{d-1}\left(\omega^2 - \frac{X}{1-X}\omega + \frac{5}{3}k^2(1-X)\right) = 0\,. \tag{8.70}$$

Here, d is the spatial dimension.

From the dispersion relation (8.70), we easily derive the following classification of the hydrodynamic modes:

(i) For $d = 1$, the spectrum of the hydrodynamic modes is purely acoustic with the dispersion ω_a which is given by (8.35):

$$\omega_a = \frac{X}{2(1-X)} \pm i\frac{k}{2}\sqrt{\frac{5X^2 - 16X + 20}{3}}\,, \tag{8.71}$$

where $X = X(k^2)$ is the real-valued root of (8.66). Since X is a negative function for all $k > 0$, the attenuation rate of the acoustic modes, $\mathrm{Re}(\omega_a)$, is negative for all $k > 0$, and the exact acoustic spectrum of the Chapman–Enskog procedure is free of the Bobylev instability for arbitrary wave lengths.

(ii) For $d > 1$, the dispersion of the acoustic modes is given by (8.71). As follows from the Chapman–Enskog procedure, the diffusion-like (real-valued) mode has the dispersion ω_{d}:

$$\omega_{\mathrm{d}} = \begin{cases} -\frac{1}{2}\left(1 - \sqrt{1 - 4k^2}\right) & k < k_{\mathrm{c}} \\ \text{none} & k > k_{\mathrm{c}} \end{cases} \tag{8.72}$$

The diffusion mode is $(d-1)$ times degenerated, the corresponding attenuation rate is negative for $k < k_{\mathrm{c}}$, and this mode *cannot be extended beyond the critical value* $k_{\mathrm{c}} = \frac{1}{2}$ *within the Chapman–Enskog method.*

The reason why this rather remarkable peculiarity of the Chapman–Enskog procedure occurs can be found upon closer investigation of the spectrum of the underlying Grad moment system (8.38).

Indeed, in the original system (8.38), besides the hydrodynamic modes, there exist several non-hydrodynamic modes which are irrelevant to the Chapman–Enskog solution. All these non-hydrodynamic modes are characterized by the property that the corresponding dispersion relations $\omega(\boldsymbol{k})$ do not go to zero, as $k \to 0$. At the point $k_{\mathrm{c}} = \frac{1}{2}$, the diffusion branch (8.72) intersects with one of the non-hydrodynamic branches of (8.38). For larger values of the wave vector k, these two branches produce a pair of complex conjugate solutions with the real part equal to $-\frac{1}{2}$. Thus, though the spectrum of the original equations (8.38)indeed continues past k_{c}, *the Chapman–Enskog method does not recognize this extension as part of the hydrodynamic branch.* It is also interesting to notice that if we would accept all the roots of (8.67), including the complex-values for $k > k_{\mathrm{c}}$, and not only the real-valued root as suggested by the asymptotics of the Chapman–Enskog solution (see the explanations preceding (8.68)), then we would come in (8.70) to the structure of the dispersion relation just mentioned.

The attenuation rates (the functions $\mathrm{Re}(\omega_{\mathrm{a}})$ and $\mathrm{Re}(\omega_{\mathrm{d}})$) are plotted in Fig. 8.3, together with the relevant dependencies for the approximations of the Chapman–Enskog method. The non-hydrodynamic branch of (8.38) which causes the breakdown of the Chapman–Enskog solution is also represented in Fig. 8.3. It is rather remarkable that while the exact hydrodynamic description becomes inapplicable for the diffusion branch at $k \geq k_{\mathrm{c}}$, the usual Navier–Stokes description still provides a good approximation to the acoustic mode around this point.

The analysis of this section leads to the following additional remarks to the conclusions made at the end of Sect. 8.3.1:

(i) The developed approach provides an understanding of the features of Chapman–Enskog solutions and the problem of extending the hydrodynamic modes into a highly non-equilibrium domain on the exact basis and in the full spatial dimension. The exact acoustic mode in the framework of the Chapman–Enskog procedure is demonstrated to be stable for all wave lengths, while the diffusion-like mode can be regarded for the hydrodynamic mode only in a bounded domain $k < k_{\mathrm{c}}$. It is remarkable that the result of the

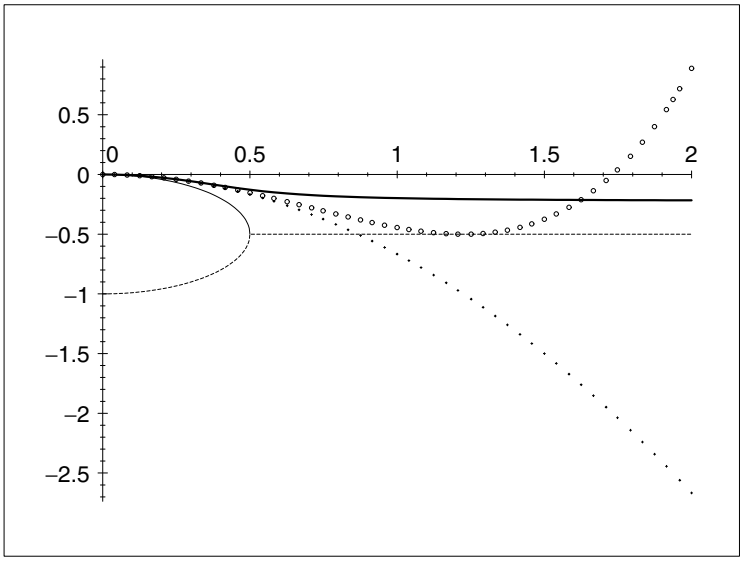

Fig. 8.3. Attenuation rates for the $3D10M$ Grad system as functions of $|k|$. *Bold*: The acoustic branch, exact summation. *Dots*: The acoustic branch, Navier–Stokes approximation. *Circles*: The acoustic branch, super-Burnett approximation. *Solid*: The diffusion branch, exact summation. *Dash*: The critical mode of the $3D10M$ Grad system

Chapman–Enskog procedure has a clear non-polynomial character. As a conjecture here, the resulting hydrodynamics is *essentially* nonlocal in space. It is also clear that *any* polynomial approximation to the Chapman–Enskog series will fail to reproduce the peculiarity of the diffusion mode demonstrated in the framework of the exact solution.

(ii) Concerning the extension of hydrodynamics into a highly non-equilibrium domain on the basis of the Boltzmann equations, the question remains open in the sense of an exact summation as above. In this respect, results for simplified models can serve either for testing approximate procedures or at least as guide. In particular, the mechanism of the singularity of the diffusion-like mode through a coupling to the non-hydrodynamic mode might be a rather general mechanism of limiting of the hydrodynamic description, and not just a feature of the Grad systems.

(iii) The result of this section demonstrates that the sum of the Chapman–Enskog series amounts to either a quite regular function (as is the function X), or to a function with a singularity at finite k_c. In both cases, however, the "smallness" of the Knudsen number ϵ used to develop the Chapman–Enskog procedure plays no role in the result of the Chapman–Enskog procedure.

8.4 The Dynamic Invariance Principle

8.4.1 Partial Summation of the Chapman–Enskog Expansion

The examples considered above demonstrate that it makes sense to speak about the sum of the Chapman–Enskog expansion, at least when the Chapman–Enskog method is applied to the (linearized) Grad equations. However, even in this case, the possibility to perform the summation exactly seems to be the lucky exception rather than the rule. Indeed, computations become more bulky with the increase of the number of the moments included in the Grad equations. Therefore, we arrive at the question: how can we approximate the recurrence equations of the Chapman–Enskog method to account for all the orders in the Knudsen number? Any such method amounts to some "partial" summation of the Chapman–Enskog expansion, and this type of working with formal series is widely spread in various fields of physics.

In this section we shall discuss a method of approximating the Chapman–Enskog expansion as a whole. As we now have the exact expressions for the Chapman–Enskog solution for the linearized 10 moment Grad equations, it is natural to start with this example for comparison purposes.

Let us come back to the originating one-dimensional Grad equations (8.10), and to the corresponding formulas of the Chapman–Enskog method (8.12) and (8.13). Instead of using the exact equations (8.12) in each order n, we introduce the following approximate equations:

Let $N \geq 1$ be some fixed integer. Then, instead of equations (8.12), we write:

$$\sigma^{(n)} = -\sum_{m=0}^{n-1} \partial_t^{(m)} \sigma^{(n-1-m)}, \quad n \leq N\,, \tag{8.73}$$

$$\sigma^{(n)} = -\sum_{m=0}^{N-1} \partial_t^{(m)} \sigma^{(n-1-m)}, \quad n > N\,. \tag{8.74}$$

This approximation amounts to the following: up to order N, the Chapman–Enskog procedure (8.12) is taken exactly (equation (8.73)), while in the computation of higher orders (equation (8.74)) we restrict the set of the Chapman–Enskog operators (8.13) only up to order N. Thus, the Chapman–Enskog coefficients $\sigma^{(n)}$ of order higher than N are taken into account only "partially". As N tends to infinity, the recurrence procedure (8.73) and (8.74) tends formally to the exact Chapman–Enskog procedure (8.12). We shall further refer to (8.73) and (8.74) as the *regularization* of the $N-th$ order. In particular, taking $N = 1$, we come to the regularization of the Burnett approximation, taking $N = 2$ we come to the regularization of the super-Burnett approximation, etc.

It can be demonstrated that the approximate procedure just described does not alter the structure of the functions $\sigma^{(2n)}$ and $\sigma^{(2n+1)}$ (8.20), while

the recurrence equations for the coefficients a_n and b_n (8.20) will differ from the exact result of the full Chapman–Enskog procedure (8.28). The advantage of the regularization procedure (8.73) and (8.74) over the exact Chapman–Enskog recurrence procedure (8.12) is that the resulting equations for the coefficients a_n and b_n are always linear, as they result from (8.73) and (8.74). This feature enables one to sum up the corresponding series exactly, even if the originating nonlinear procedure leads to a too difficult analysis. The number N can be called the "depth" of the approximation: the large N is, the more low-order terms of the Chapman–Enskog expansion are taken into account exactly due to (8.73).

For the first example, let us take $N = 1$ in (8.73) and (8.74). The regularization of the Burnett approximation then reads:

$$\sigma^{(n)} = -\partial_t^{(0)} \sigma^{(n-1)} \,, \tag{8.75}$$

where $n \geq 1$, and $\sigma^{(0)} = -(4/3)\partial_x u$. Turning to the Fourier variables, we derive:

$$\begin{aligned} \sigma_k^{(2n)} &= a_n (-k^2)^n iku_k \,, \\ \sigma_k^{(2n+1)} &= b_n (-k^2)^{n+1} p_k \,, \end{aligned} \tag{8.76}$$

where the coefficients a_n and b_n are due to the following recurrence procedure:

$$a_{n+1} = \frac{5}{3} b_n, \quad b_n = a_n, \quad a_0 = -\frac{4}{3} \,, \tag{8.77}$$

whereupon

$$a_n = b_n = \left(\frac{5}{3}\right)^n a_0 \,. \tag{8.78}$$

Thus, denoting as σ_{1k}^R the Fourier transform of the regularized Burnett approximation, we obtain:

$$\sigma_{1k}^R = -\frac{4}{3 + 5k^2} \left(iku_k - k^2 p_k\right) \,. \tag{8.79}$$

It should be noted that the recurrence equations (8.77) can also be obtained from the exact recurrence equations (8.28) by neglecting the nonlinear terms. Thus, the approximation adopted within the regularization procedure (8.75) amounts to the following rational approximation of the functions A and B (8.22):

$$A_1^R = B_1^R = -\frac{4}{3 + 5k^2} \,. \tag{8.80}$$

Substituting the latter expressions instead of the functions A and B in the dispersion formula (8.23), we come to the dispersion relation of the hydrodynamic modes within the regularized Burnett approximation:

$$\omega_{\pm} = -\frac{2k^2}{3+5k^2} \pm i|k|\sqrt{\frac{75k^2k^2+66k^2+15}{25k^2k^2+30k^2+9}} \,. \tag{8.81}$$

The dispersion relation (8.81) is stable for all wave vectors, and in the short-wave limit we have:

$$\lim_{|k|\to\infty} \omega_{\pm} = -0.4 \pm i|k|\sqrt{3} \,. \tag{8.82}$$

Thus, the regularized Burnett approximation leads qualitatively to the same behavior of the dispersion relation, as the exact result (8.36), with the limiting value of the attenuation rate equal to -0.4 instead of the exact value $-2/9$.

Consider now the regularization of the super-Burnett approximation. This amounts to setting $N = 2$ in the recurrence equations (8.73) and (8.74). Then, instead of (8.75), we have:

$$\begin{aligned} \sigma^{(1)} &= -\partial_t^{(0)}\sigma^{(0)} \,, \\ \sigma^{(2+n)} &= -\partial_t^{(0)}\sigma^{(n+1)} - \partial_t^{(1)}\sigma^{(n)} \,, \end{aligned} \tag{8.83}$$

where $n \geq 0$. The corresponding recurrence equations for the coefficients a_n and b_n now become:

$$a_{n+1} = \frac{1}{3}b_n, \quad a_n = b_n, \quad a_0 = -\frac{4}{3} \,. \tag{8.84}$$

Thus, instead of (8.80), we obtain:

$$A_2^R = B_2^R = -\frac{4}{3+k^2} \,. \tag{8.85}$$

The corresponding dispersion relation of the regularized super-Burnett approximation reads:

$$\omega_{\pm} = -\frac{2k^2}{3+k^2} \pm i|k|\sqrt{\frac{25k^2k^2+78k^2+45}{3k^2k^2+18k^2+27}} \,, \tag{8.86}$$

while in the short-wave limit the asymptotic behavior becomes:

$$\lim_{|k|\to\infty} \omega_{\pm} = -2 \pm i|k|\sqrt{\frac{25}{3}} \,. \tag{8.87}$$

The Bobylev instability is removed again within the regularization of the super-Burnett approximation, and the lower-order terms of the Chapman–Enskog expansion are taken into account more precisely in comparison to the regularized Burnett approximation. However, the approximation in a whole has not improved (see Fig. 8.4). *Thus, we can conclude that although the partial summation method* (8.73) *and* (8.74) *is capable of removing the Bobylev instability, and reproducing qualitatively the exact Chapman–Enskog solution*

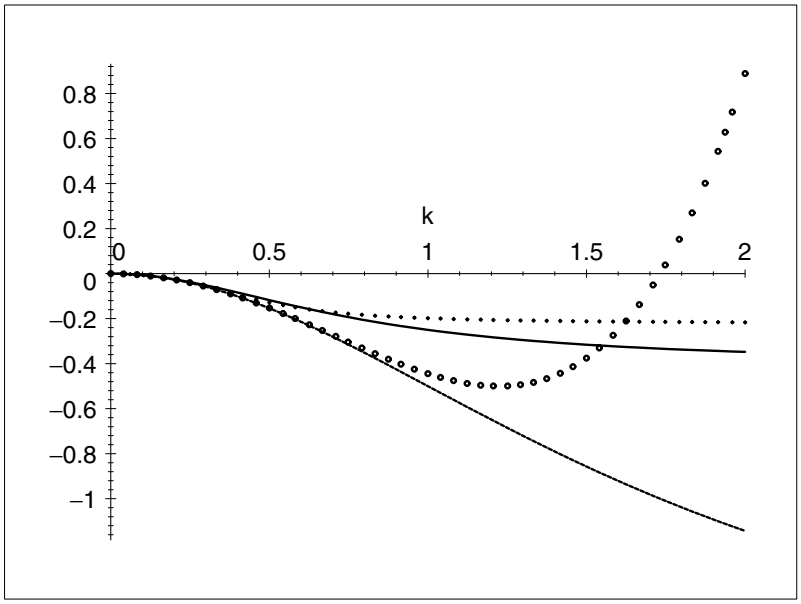

Fig. 8.4. Attenuation rates for the partial summing. *Solid*: The regularized Burnett approximation. *Dash*: The regularized super-Burnett approximation. *Circles*: The super-Burnett approximation. *Dots*: The exact summation

in the short-wave domain, the exactness does not increase monotonically with the depth of the approximation N. This drawback of the regularization procedure indicates once again that an attempt to capture the lower-order terms of the Chapman–Enskog procedure does not succeed in a better approximation as a whole.

8.4.2 The Dynamic Invariance

The starting points of all the approaches considered so far (exact or approximate) is the Chapman–Enskog expansion. However, the result of the summation does not involve the Knudsen number ϵ explicitly and does not require the "smallness" of this parameter. Therefore, it makes sense to reformulate the problem of the reduced description (for the Grad equations (8.10) this amounts to the problem of constructing a function $\sigma_k(u_k, p_k, k)$) in a way where the parameter ϵ does not appear at all. Further, in the framework of such an approach, we can seek a method of explicit construction of the function $\sigma_k(u_k, p_k, k)$, which does not rely upon the Taylor-like expansions as above.

In this section we introduce such an approach, considering again the illustrative example (8.10). These ideas will be extensively used in the sequel, and

they also constitute the basis of the so-called method of invariant manifold for dissipative systems [11].

Let us rewrite here (8.10) in the Fourier variables, and cancel the parameter ϵ:

$$\begin{aligned} \partial_t p_k &= -\frac{5}{3} i k u_k \ , \\ \partial_t u_k &= -ik p_k - ik\sigma_k \ , \\ \partial_t \sigma_k &= -\frac{4}{3} ik u_k - \sigma_k \ . \end{aligned} \tag{8.88}$$

The result of the reduction in the system (8.88) amounts to a function $\sigma_k(u_k, p_k, k)$, which depends parametrically on the hydrodynamic variables u_k and p_k, and also on the wave vector k. Due to the linearity of the problem under consideration, this function depends linearly on u_k and p_k, and we can start with the form given by (8.21):

$$\sigma_k(u_k, p_k, k) = ikAu_k - k^2 B p_k \ , \tag{8.89}$$

where A and B are undetermined functions of k. Now, however, we do not refer to a power series representation of these functions as in (8.22).

Given the form of the function $\sigma_k(u_k, p_k, k)$ (8.89), we can compute its time derivative in *two* different ways. On one hand, substituting (8.89) into the right hand side of the third equation in the set (8.88), we derive:

$$\partial_t^{\text{micro}} \sigma_k = -ik \left(\frac{4}{3} + A \right) u_k + k^2 B p_k \ . \tag{8.90}$$

On the other hand, computing the time derivative and using the first two equations (8.88), we obtain:

$$\begin{aligned} \partial_t^{\text{macro}} \sigma_k &= \frac{\partial \sigma_k}{\partial u_k} \partial_t u_k + \frac{\partial \sigma_k}{\partial p_k} \partial_t p_k \\ &= ikA \left(-ikp_k - ik\sigma_k \right) - k^2 B \left(-\frac{5}{3} ik u_k \right) \\ &= ik \left(\frac{5}{3} k^2 B + k^2 A \right) u_k + k^2 \left(A - k^2 B \right) p_k \ . \end{aligned} \tag{8.91}$$

Equating the expressions in the right hand sides of (8.90) and (8.91), and requiring that the resulting equality holds for any values of the variables u_k and p_k, we derive the following two algebraic equations:

$$\begin{aligned} F(A, B, k) &= -A - \frac{4}{3} - k^2 \left(\frac{5}{3} B + A^2 \right) = 0 \ , \\ G(A, B, k) &= -B + A \left(1 - k^2 B \right) = 0 \ . \end{aligned} \tag{8.92}$$

These are exactly the equations (8.32), which were obtained after summation of the Chapman–Enskog expansion. Now, however, we have reached the same

result without using the expansion. Thus, (8.92) (or, equivalently, (8.32)) can be used as a starting point for the construction of the function (8.89).

It is important to comment on the somewhat formal manipulations which have led to (8.92). First of all, by the very sense of the reduced description problem, we are looking for a set of functions σ_k which depend on time only through the time dependence of the hydrodynamic variables u_k and p_k. That is, we are looking for a set (8.89), which is parameterized with the values of the hydrodynamic variables. Further, the two time derivatives, (8.90) and (8.91), are relevant to the "microscopic" and the "macroscopic" evolution within the set (8.89), respectively. Indeed, the expression in the right hand side of (8.90) is just the value of the vector field of the original Grad equations at the points of the set (8.89). On the other hand, (8.91) expresses the time derivative in terms of the reduced (macroscopic) dynamics, which, in turn, is self-consistently defined by the form (8.89). Equations (8.92) provide, therefore, the *dynamic invariance condition of the reduced description* for the set (8.89): the function $\sigma_k(u_k(t), p_k(t), k)$ is a solution to both the full Grad system (8.88) and to the reduced system which consists of the first two (hydrodynamic) equations. For this reason, equations (8.92) and their analogs which will be obtained on similar reasoning, will be called *the invariance equations*.

8.4.3 The Newton Method

Let us concentrate on the problem of solving the invariance equations (8.92). Clearly, if we are going to expand the functions A and B into power series (8.22), we shall return to the Chapman–Enskog procedure. Now, however, we see that the Chapman–Enskog expansion is just a method to solve the invariance equations (8.92), and maybe not even the optimal one.

Another possibility is to use *iterative* methods. Indeed, we shall apply Newton's method. The algorithm is as follows: Let A_0 and B_0 are some initial approximations chosen for the procedure. The correction, $A_1 = A_0 + \delta A_1$ and $B_1 = B_0 + \delta B_1$, due to the Newton iteration is obtained upon a linearization (8.92) around the approximation A_0 and B_0. Computing the derivatives, we can represent the equation of the Newton iteration in matrix form:

$$\begin{pmatrix} \frac{\partial F(A,B,k)}{\partial A}\,|_{A=A_0,B=B_0} & \frac{\partial F(A,B,k)}{\partial B}\,|_{A=A_0,B=B_0} \\ \frac{\partial G(A,B,k)}{\partial A}\,|_{A=A_0,B=B_0} & \frac{\partial G(A,B,k)}{\partial B}\,|_{A=A_0,B=B_0} \end{pmatrix} \begin{pmatrix} \delta A_1 \\ \delta B_1 \end{pmatrix} + \begin{pmatrix} F(A_0,B_0,k) \\ G(A_0,B_0,k) \end{pmatrix} = 0\,. \tag{8.93}$$

where

$$\begin{aligned} \frac{\partial F(A,B,k)}{\partial A}\,|_{A=A_0,B=B_0} &= -\left(1+2k^2A_0\right)\,, \\ \frac{\partial F(A,B,k)}{\partial B}\,|_{A=A_0,B=B_0} &= -\frac{5}{3}k^2\,, \end{aligned} \tag{8.94}$$

$$\frac{\partial G(A,B,k)}{\partial A}\,|_{A=A_0,B=B_0} = 1 - k^2 B_0 \ ,$$
$$\frac{\partial G(A,B,k)}{\partial B}\,|_{A=A_0,B=B_0} = -\left(1 + k^2 A_0\right) \ .$$

Solving the system of linear algebraic equations, we come to the first correction δA_1 and δB_1. Further corrections are found iteratively:

$$A_{n+1} = A_n + \delta A_{n+1} \ , \qquad (8.95)$$
$$B_{n+1} = B_n + \delta B_{n+1} \ ,$$

where $n \geq 0$, and

$$\begin{pmatrix} -(1+2k^2 A_n) & -\frac{5}{3}k^2 \\ 1 - k^2 B_n & -(1+k^2 A_n) \end{pmatrix} \begin{pmatrix} \delta A_{n+1} \\ \delta B_{n+1} \end{pmatrix} + \begin{pmatrix} F(A_n, B_n, k) \\ G(A_n, B_n, k) \end{pmatrix} = 0 \ . \quad (8.96)$$

Within the algorithm just presented, the problem is how to choose the initial approximation A_0 and B_0. The recursion (8.95) and (8.96) is applicable formally to any initial approximation. However, the convergence (if at all) might be sensitive to the choice.

For the first experiment let us take the Navier–Stokes approximation of the functions A and B:

$$A_0 = B_0 = -\frac{4}{3}$$

The outcome of the first two Newton iterations (the attenuation rates as they follow from the first and second Newton iteration) are presented in Fig. 8.5. It is clearly seen that the Newton iterations converge rapidly to the exact solution for moderate k, but the asymptotic behavior in the short-wave domain does not improve.

Another possibility is to take the result of the regularization procedure as presented above. Let the regularized Burnett approximation (8.80) be taken for the initial approximation, that is:

$$A_0 = A_1^R = -\frac{4}{3+5k^2} \ , \quad B_0 = B_1^R = -\frac{4}{3+5k^2} \ . \qquad (8.97)$$

Substituting (8.97) into (8.95) and (8.96) for $n = 0$ we obtain, after some algebra, the following first correction:

$$A_1 = -\frac{4(27 + 63k^2 + 153k^2k^2 + 125k^2k^2k^2)}{3(3+5k^2)(9+9k^2+67k^2k^2+75k^2k^2k^2)} \ , \qquad (8.98)$$
$$B_1 = -\frac{4(9 + 33k^2 + 115k^2k^2 + 75k^2k^2k^2)}{(3+5k^2)(9+9k^2+67k^2k^2+75k^2k^2k^2)}$$

Functions (8.98) are not yet the exact solution to (8.92) (that is, the functions $F(A_1, B_1, k)$ and $G(A_1, B_1, k)$ are not equal to zero for all k). However,

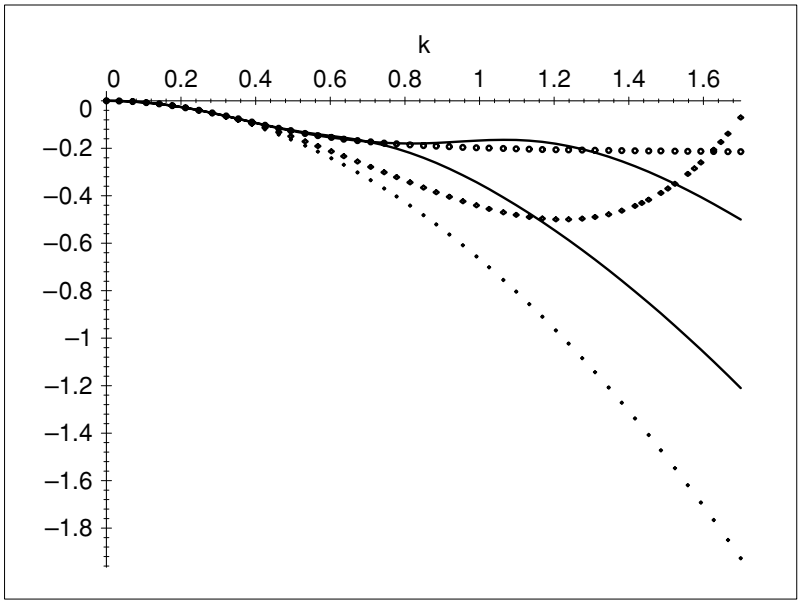

Fig. 8.5. Attenuation rates for the Newton method with the Navier–Stokes approximation as the initial condition. *Dots*: The Navier–Stokes approximation. *Solid*: The first and the second iterations of the invariance equation. *Circles*: The exact solution to the invariance equation. *Diamonds*: The super-Burnett approximation

substituting A_1 and B_1 instead of A and B into the dispersion relation (8.23), we derive in the short-wave limit:

$$\lim_{|k|\to\infty} \omega_{\pm} = -\frac{2}{9} \pm i|k|\sqrt{3}\,. \tag{8.99}$$

That is, already the first Newton iteration, as applied to the regularized Burnett approximation, leads to the exact expression in the short-wave domain. Since the first Newton iteration appears to be asymptotically exact, the next iterations improve the solution only for the intermediate values of k, whereas the asymptotic behaviour remains exact in all iterations. The attenuation rates for the first and second Newton iterations with the initial approximation (8.97) are plotted in Fig. 8.6. The agreement with the exact solution is excellent.

One more test is to take the result of the super-Burnett approximation (8.85) as an initial condition in the Newton procedure (8.96). As we know, the regularization of the super-Burnett approximation provides a poorer approximation in comparison to (8.97), particularly in the short-wave domain. Nevertheless, the Newton iterations do converge though less rapidly (see Fig. 8.7).

The examples considered so far demonstrate that the Newton method, as applied to the invariance equations (8.92) is a more powerful tool in

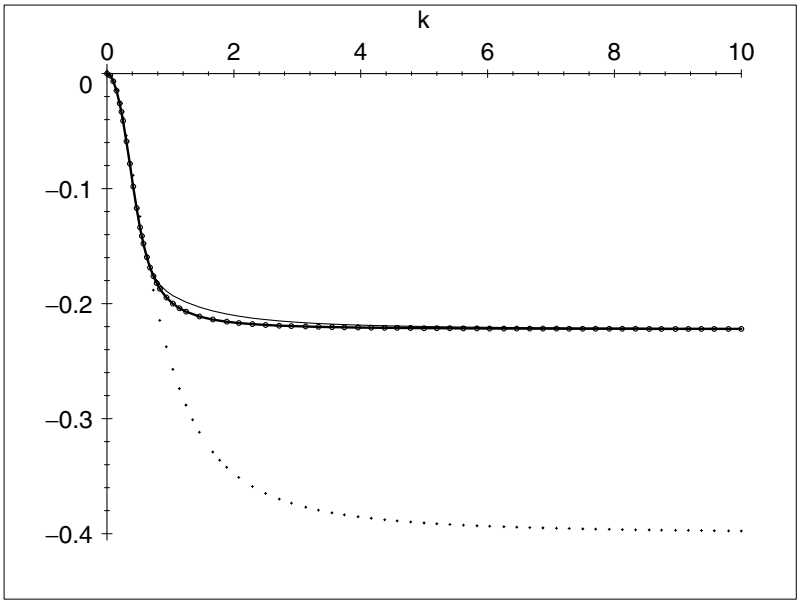

Fig. 8.6. Attenuation rates with the regularized Burnett approximation as the initial condition for the Newton method. *Dots*: The regularized Burnett approximation, or the first Newton iteration with the Euler initial condition (see text). *Solid*: The first and the second Newton iterations with the regularized Burnett approximation as the initial condition. *Circles*: The exact solution to the invariance equation

comparison to the Chapman–Enskog procedure. It is also important that the initial approximation should be "properly chosen", and that it should reproduce, at least qualitatively, the features of the solution not only in the long-wave limit, but over the whole range of wavenumbers.

The best from the initial approximations considered so far is the regularized Burnett approximation (8.97). We have already commented on the relation of this approximation to the invariance equations, as well as on its relation to the Chapman–Enskog procedure. The further important observation is as follows:

Let us choose the *Euler* approximation for the functions A and B, that is:

$$A_0 = B_0 = 0 \tag{8.100}$$

The equation of the first Newton iteration (8.96) is very simple:

$$\begin{pmatrix} -1 & -\frac{5}{3}k^2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \delta A_1 \\ \delta B_1 \end{pmatrix} + \begin{pmatrix} -\frac{4}{3} \\ 0 \end{pmatrix} = 0 \,, \tag{8.101}$$

and

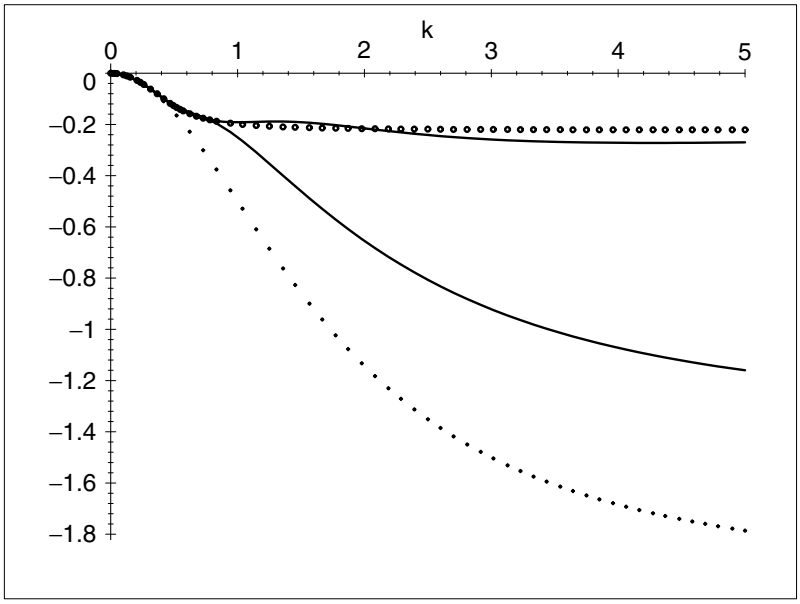

Fig. 8.7. Attenuation rates with the regularized super-Burnett approximation as the initial condition for the Newton method. *Dots*: The regularized super-Burnett approximation. *Solid*: The first and the second Newton iterations. *Circles*: The exact solution to the invariance equation

$$A_1 = B_1 = -\frac{4}{3 + 5k^2} \ . \tag{8.102}$$

Thus, *the regularized Burnett approximation is at the same time the first Newton correction as applied to the Euler initial approximation*. This property distinguishes the regularization of the Burnett approximation from other regularizations. Now the functions (8.98) can be regarded as the *second* Newton correction as applied to the Euler initial approximation (8.100).

Finally, let us examine what Newton's method does in the case of singularities. As we have demonstrated in the previous section, the singularity of the diffusion-like mode occurs when this mode couples to a non-hydrodynamic mode of the 10 moment Grad system if the spatial dimension is greater that one.

Without proving it here, the invariance equation method as applied to the 10 moment Grad system (8.38) leads to the system of equations (8.65). We have already demonstrated what the outcome of the Newton method is when it is applied to the first two equations of this system (responsible for the acoustic mode and containing no singularities). The Newton method, as applied to (8.67), reads:

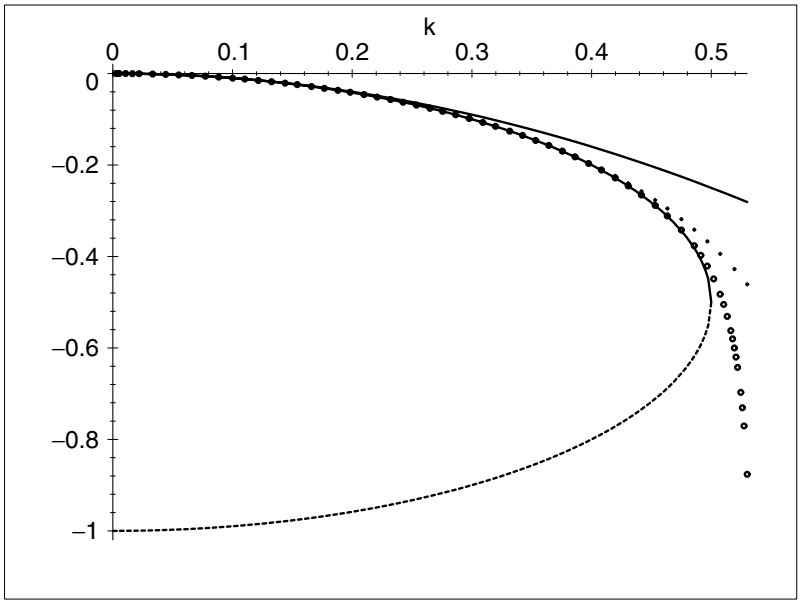

Fig. 8.8. The diffusion mode with the Euler initial approximation for the invariance equation. *Upper solid*: The the first iteration. *Dots*: The second iteration. *Circles*: The third iteration. *Lower solid*: The exact solution. *Dash*: The critical mode

$$
\begin{aligned}
&Y_{n+1} = Y_n + \delta Y_{n+1} \,, \\
&(1 + 2Y_n)\delta Y_{n+1} + \{Y_n(1 + Y_n) + k^2\} = 0 \,,
\end{aligned}
\tag{8.103}
$$

where $n \geq 0$, and Y_0 is a chosen initial approximation. Taking the Euler approximation ($Y_0 = 0$), we derive:

$$
\begin{aligned}
Y_1 &= -k^2 \,, \\
Y_2 &= -\frac{k^2(1 + k^2)}{1 - 2k^2} \,.
\end{aligned}
\tag{8.104}
$$

The second approximation, Y_2, is singular at $k_2 = \sqrt{1/2}$, and it can be demonstrated that all further corrections also have the first singularity at points k_n, and the sequence $k_2, \ldots, k_n$ tends to the actual branching point of the invariance equation (8.67) $k_c = 1/2$. The analysis of further corrections demonstrates that the convergence is very rapid (see Fig. 8.8).

The expressions (8.104) demonstrate that unlike polynomial approximations, Newton's method is capable of detecting the actual singularities of the hydrodynamic spectrum. Formally, the function Y_2 becomes positive as k becomes larger than k_2, and thus the attenuation rate, $\omega_d = Y_2$ becomes positive after this point. However, unlike the super-Burnett approximation for the acoustic mode, this transition occurs now at a singular point. Indeed, the

attenuation rate Y_2 tends to "minus infinity", as k tends to k_2 from the left. Thus, as described with the Newton procedure, the non-physical domain is separated from the physical one with an "infinitely viscid" threshold. The occurrence of the poles in the Newton iterations is, of course, quite clear. Indeed, the Newton method involves the derivative of the function $R(Y) = Y(Y+1)$ which appears on the left hand side of (8.67). The derivative $dR(Y)/dY$ becomes zero at the singularity point $Y_c = -1/2$. The results of this section bring us to the following conclusion:

(i) Exact summation of the Chapman–Enskog procedure results in the same system of equations as the principle of dynamic invariance. This was demonstrated above for a specific situation but it holds for any (linearized) Grad system. The resulting equations are always *nonlinear* (even for the simplest linearized kinetic systems, such as Grad equations).

(ii) Now we are able to *alter the viewpoint*: the invariance equations can be considered as basic in the theory, while the Chapman–Enskog method is a way to solve it via an expansion in powers of k. The method of power series expansion is neither the only method to solve equations, nor the optimal. Alternative iteration methods might be better suited to the problem of constructing the reduced description.

(iii) An opportunity to derive the invariance equation in closed form, and next to solve it this or that way is, of course, rather exotic. The situation becomes complicated already for the nonlinear Grad equations, and we should not expect anything simple in the case of the Boltzmann equation. Therefore, if we are willing to proceed along these lines in other problems, attention should be drawn towards approximate procedures. With this, the question arises: what amount of information is required to execute the procedures? Indeed, the Navier–Stokes approximation can be obtained without any knowledge of the whole nonlinear system of invariance equations. It is important that the Newton method, as applied to our problem, does not require any global information as well. This was demonstrated above by a relation between the first iteration as applied to the Euler approximation and the regularization of the Burnett approximation.

8.4.4 Invariance Equation for the $1D13M$ Grad System

Let us consider as the next example the problem of the reduced description for the one-dimensional thirteen moment Grad system. Using the dimensionless variables as above, we write the one-dimensional version of the Grad equations (8.2) and (8.3) in the k-representation:

$$\begin{aligned} \partial_t \rho_k &= -iku_k \,, \\ \partial_t u_k &= -ik\rho_k - ikT_k - ik\sigma_k \,, \\ \partial_t T_k &= -\frac{2}{3}iku_k - \frac{2}{3}ikq_k \,, \end{aligned} \tag{8.105}$$

$$\partial_t \sigma_k = -\frac{4}{3} iku_k - \frac{8}{15} ikq_k - \sigma_k \ ,$$
$$\partial_t q_k = -\frac{5}{2} ikT_k - ik\sigma_k - \frac{2}{3} q_k \ .$$

The Grad system (8.105) provides the simplest coupling of the hydrodynamic variables ρ_k, u_k, and T_k to the non-hydrodynamic variables, σ_k and q_k, the latter corresponding to the heat flux. As above, our goal is to reduce the description of the Grad system (8.105) to the three hydrodynamic equations with respect to the variables ρ_k, u_k, and T_k. That is, we have to express the functions σ_k and q_k in terms of ρ_k, u_k, and T_k:

$$\sigma_k = \sigma_k(\rho_k, u_k, T_k, k) \ ,$$
$$q_k = q_k(\rho_k, u_k, T_k, k) \ .$$

Application of the Chapman–Enskog method in these cases, results in the following algebraic scheme (we omit the Knudsen number ϵ):

$$\sigma_k^{(n)} = -\left\{ \sum_{m=0}^{n-1} \partial_t^{(m)} \sigma_k^{(n-1-m)} + \frac{8}{15} ikq_k^{(n-1)} \right\} \ , \tag{8.106}$$
$$q_k^{(n)} = -\left\{ \sum_{m=0}^{n-1} \partial_t^{(m)} q_k^{(n-1-m)} + ik\sigma_k^{(n-1)} \right\} \ ,$$

where the Chapman–Enskog operators act as follows:

$$\partial_t^{(m)} \rho_k = \begin{cases} -iku_k, & m = 0 \\ 0, & m \geq 1 \end{cases} \ , \tag{8.107}$$
$$\partial_t^{(m)} u_k = \begin{cases} -ik(\rho_k + T_k), & m = 0 \\ -ik\sigma_k^{(m-1)}, & m \geq 1 \end{cases} \ ,$$
$$\partial_t^{(m)} T_k = \begin{cases} -\frac{2}{3} iku_k, & m = 0 \\ -\frac{2}{3} ikq_k^{(m-1)}, & m \geq 1 \end{cases} \ .$$

The initial condition for the recurrence procedure (8.106) reads: $\sigma_k^{(0)} = -\frac{4}{3} iku_k$, and $q_k^{(0)} = -\frac{15}{4} ikT_k$, which leads to the Navier–Stokes-Fourier hydrodynamic equations.

Computing the coefficients $\sigma_k^{(1)}$ and $q_k^{(1)}$, obtain the Burnett approximation:

$$\sigma_{1k} = -\frac{4}{3} iku_k + \frac{4}{3} k^2 \rho_k - \frac{2}{3} k^2 T_k \ , \tag{8.108}$$
$$q_{1k} = -\frac{15}{4} ikT_k + \frac{7}{4} k^2 u_k \ .$$

The Burnett approximation (8.108) coincides with that obtained from the Boltzmann equation, and it is precisely the case where the instability was first demonstrated in the paper [72].

The structure of the terms $\sigma_k^{(n)}$ and $q_k^{(n)}$ (an analog of (8.20) and (8.42)) is as follows:

$$\begin{aligned}
\sigma_k^{(2n)} &= a_n(-k^2)^n iku_k \ , \\
\sigma_k^{(2n+1)} &= b_n(-k^2)^{n+1}\rho_n + c_n(-k^2)^{n+1}T_k \ , \\
q_k^{(2n)} &= \beta_n(-k^2)^n ik\rho_k + \gamma_n(-k^2)^n ikiT_k \ , \\
q_k^{(2n+1)} &= \alpha_n(-k^2)^{n+1}u_k \ .
\end{aligned} \tag{8.109}$$

The derivation of the invariance equation for the system (8.105) goes along the same lines as in the previous section. We seek the functions of the reduced description in the form:

$$\begin{aligned}
\sigma_k &= ikAu_k - k^2B\rho_k - k^2CT_k \ , \\
q_k &= ikX\rho_k + ikYT_k - k^2Zu_k \ ,
\end{aligned} \tag{8.110}$$

where the functions $A, \ldots, Z$ will be determinated in the process.

The invariance condition results in a closed system of equations for the functions A, B, C, X, Y, and Z. As above, computing the microscopic time derivative of the functions (8.110), due to the two last equations of the Grad system (8.105) we derive:

$$\begin{aligned}
\partial_t^{\text{micro}}\sigma_k &= -ik\left(\frac{4}{3} - \frac{8}{15}k^2Z + A\right)u_k \\
&\quad + k^2\left(\frac{8}{15}X + B\right)\rho_k + k^2\left(\frac{8}{15}Y + C\right)T_k \ , \\
\partial_t^{\text{micro}}q_k &= k^2\left(A + \frac{2}{3}Z\right)u_k + ik\left(k^2B - \frac{2}{3}X\right)\rho_k \\
&\quad - ik\left(\frac{5}{2} - k^2C - \frac{2}{3}Y\right)T_k \ .
\end{aligned} \tag{8.111}$$

On the other hand, computing the macroscopic time derivative due to the first three equations of the system (8.105), we obtain:

$$\begin{aligned}
\partial_t^{\text{macro}}\sigma_k &= \frac{\partial\sigma_k}{\partial u_k}\partial_t u_k + \frac{\partial\sigma_k}{\partial\rho_k}\partial_t\rho + \frac{\partial\sigma_k}{\partial T_k}\partial_t T_k \\
&= ik\left(k^2A^2 + k^2B + \frac{2}{3}k^2C - \frac{2}{3}k^2k^2CZ\right)u_k \\
&\quad + \left(k^2A - k^2k^2AB - \frac{2}{3}k^2k^2CX\right)\rho_k \\
&\quad + \left(k^2A - k^2k^2AC - \frac{2}{3}k^2k^2CY\right)T_k \ ; \\
\partial_t^{\text{macro}}q_k &= \frac{\partial q_k}{\partial u_k}\partial_t u_k + \frac{\partial q_k}{\partial\rho_k}\partial_t\rho u_k + \frac{\partial q_k}{\partial T_k}\partial_t T_k
\end{aligned} \tag{8.112}$$

$$= \left(-k^2k^2ZA + k^2X + \frac{2}{3}k^2Y - \frac{2}{3}k^2k^2YZ\right) u_k$$
$$+ik\left(k^2Z - k^2k^2ZB + \frac{2}{3}k^2YX\right)\rho_k$$
$$+ik\left(k^2Z - k^2k^2ZC + \frac{2}{3}k^2Y^2\right)T_k \ .$$

Equating the corresponding expressions in the formulas (8.111) and (8.112), we derive the following system of coupled equations:

$$\begin{aligned}
F_1 &= -\frac{4}{3} + \frac{8}{15}k^2Z - A - k^2A^2 - k^2B - \frac{2}{3}k^2C + \frac{2}{3}k^2k^2CZ = 0 \ , \\
F_2 &= \frac{8}{15}X + B - A + k^2AB + \frac{2}{3}k^2CX = 0 \ , \\
F_3 &= \frac{8}{15}Y + C - A + k^2AC + \frac{2}{3}k^2CY = 0 \ , \\
F_4 &= A + \frac{2}{3}Z + k^2ZA - X - \frac{2}{3}Y + \frac{2}{3}k^2YZ = 0 \ , \\
F_5 &= k^2B - \frac{2}{3}X - k^2Z + k^2k^2ZB - \frac{2}{3}k^2YX = 0 \ , \\
F_6 &= -\frac{5}{2} + k^2C - \frac{2}{3}Y - k^2Z + k^2k^2ZC - \frac{2}{3}k^2Y^2 = 0 \ .
\end{aligned} \tag{8.113}$$

As above, the invariance equations (8.113) can also be obtained upon summation of the Chapman–Enskog expansion, after the Chapman–Enskog procedure is casted into a recurrence relations for the coefficients $a_n, \ldots, \alpha_n$ (8.109). This route is less straightforward than the one just presented, and we omit the proof.

The Newton method, as applied to the system (8.113), results in the following algorithm:

Denote as $\boldsymbol{A}$ the six-component vector function $\boldsymbol{A} = (A, B, C, X, Y, Z)$. Let $\boldsymbol{A}_0$ is the initial approximation, then:

$$\boldsymbol{A}_{n+1} = \boldsymbol{A}_n + \delta\boldsymbol{A}_{n+1} \ , \tag{8.114}$$

where $n \geq 0$, and the vector function $\delta\boldsymbol{A}_{n+1}$ is a solution to the linear system of equations:

$$\boldsymbol{N}_n\delta\boldsymbol{A}_{n+1} + \boldsymbol{F}_n = 0 \ . \tag{8.115}$$

Here $\boldsymbol{F}_n$ is the vector function with the components $F_i(\boldsymbol{A}_n)$, and $\boldsymbol{N}_n$ is a 6×6 matrix:

$$\begin{pmatrix}
-(1 + 2k^2A_n) & -k^2 & -2/3k^2(1 - k^2Z_n) \\
k^2B_n - 1 & 1 + k^2 & 2/3k^2X_n \\
k^2C_n - 1 & 0 & 1 + 2/3k^2Y_n + k^2A_n \\
1 + k^2Z_n & 0 & 0 \\
0 & k^2(1 + k^2Z_n) & 0 \\
0 & 0 & k^2(1 + k^2Z_n)
\end{pmatrix} \tag{8.116}$$

$$\begin{pmatrix} 0 & 0 & 2/3k^2(4/5+k^2C_n) \\ 2/3(4/5+k^2C_n) & 0 & 0 \\ 0 & 2/3(4/5+k^2C_n) & 0 \\ -1 & -2/3(1-k^2Z_n) & 2/3+k^2A_n+2/3k^2Y_n \\ -2/3(1+k^2Y_n) & -2/3k^2X_n & -k^2(1-k^2B_n) \\ 0 & -2/3(1+2k^2Y_n) & -k^2(1-k^2C_n) \end{pmatrix} .$$

The Euler approximation gives: $A_0 = \ldots = Z_0 = 0$, while $F_1 = -4/3$, $F_6 = -5/2$, and $F_2 = \ldots = F_5 = 0$. The first Newton iteration (8.115) as applied to this initial approximation, leads again to a simple algebraic problem, and we have finally obtained:

$$\begin{aligned} A_1 &= -20\frac{141k^2+20}{867k^4+2105k^2+300} , \\ B_1 &= -20\frac{459k^2k^2+810k^2+100}{3468k^2k^2k^2+12755k^2k^2+11725k^2+1500} , \\ C_1 &= -10\frac{51k^2k^2-485k^2-100}{3468k^2k^2k^2+12755k^2k^2+11725k^2+1500} , \\ X_1 &= -\frac{375k^2(21k^2-5)}{2(3468k^2k^2k^2+12755k^2k^2+11725k^2+1500)} , \\ Y_1 &= -\frac{225(394k^2k^2+685k^2+100)}{4(3468k^2k^2k^2+12755k^2k^2+11725k^2+1500)} , \\ Z_1 &= -15\frac{153k^2+35}{867k^4+2105k^2+300} . \end{aligned} \tag{8.117}$$

Substituting (8.109) into the first three equations of the Grad system (8.105), and proceeding with the dispersion relation as above, we derive the latter in terms of the functions $A, \ldots, Z$:

$$\begin{aligned} &\omega^3 - k^2\left(\frac{2}{3}Y+A\right)\omega^2 \\ &\quad + k^2\left(\frac{5}{3}-\frac{2}{3}k^2Z-\frac{2}{3}k^2C-k^2B+\frac{2}{3}k^2AY+\frac{2}{3}k^2k^2CZ\right)\omega \\ &\quad + \frac{2}{3}k^2(k^2X-k^2Y+k^2k^2BY-k^2k^2XC) = 0 . \end{aligned} \tag{8.118}$$

When the functions $A_1, \ldots, Z_1$ (8.117) are substituted instead of $A, \ldots, Z$ into (8.118), the dispersion relation of the first Newton iteration, as applied to the invariance equations (8.113) with the Euler initial approximation, is obtained. This result coincides with the regularization of the Burnett approximation, which was considered in [43]. There it was demonstrate that the equilibrium is stable within this approximation for arbitrary wave lengths. The dispersion relation for the Burnett approximation, in turn, is due to the approximation

$$A = -4/3\,, \quad B = -4/3\,, \quad C = 2/3\,, \quad X = 0\,, \quad Y = -15/4\,, \quad Z = -7/4\,,$$

as it follows from a comparison of (8.108) and (8.110). The dispersion relation for the Burnett approximation coincides with the one obtained in [72] from the Boltzmann equation.

8.4.5 Invariance Equation for the $3D13M$ Grad System

The final example to be considered is the 13 moment Grad system in three spatial dimensions, (8.2) and (8.3). Let us rewrite here the original system in terms of Fourier variables:

$$\begin{aligned}
\partial_t \rho_k &= -ik\boldsymbol{e}_k \cdot \boldsymbol{u}_k\,, \\
\partial_t \boldsymbol{u}_k &= -ik\boldsymbol{e}_k \rho_k - ik\boldsymbol{e}_k T_k - ik\boldsymbol{e}_k \cdot \boldsymbol{\sigma}_k\,, \\
\partial_t T_k &= -\frac{2}{3} ik(\boldsymbol{e}_k \cdot \boldsymbol{u}_k + \boldsymbol{e}_k \cdot \boldsymbol{q}_k)\,, \\
\partial_t \boldsymbol{\sigma}_k &= -ik\overline{\boldsymbol{e}_k \boldsymbol{u}_k} - \frac{2}{5} ik\overline{\boldsymbol{e}_k \boldsymbol{q}_k} - \boldsymbol{\sigma}_k\,, \\
\partial_t \boldsymbol{q}_k &= -\frac{5}{2} ik\boldsymbol{e}_k T_k - ik\boldsymbol{e}_k \cdot \boldsymbol{\sigma}_k - \frac{2}{3}\boldsymbol{q}_k\,.
\end{aligned} \tag{8.119}$$

Here we have represented the wave vector $\boldsymbol{k}$ as $\boldsymbol{k} = k\boldsymbol{e}_k$, and $\boldsymbol{e}_k$ is the unit vector.

The structure of the even and odd Chapman–Enskog coefficients, $\boldsymbol{\sigma}_k^{(n)}$ and $\boldsymbol{q}_k^{(n)}$, turns out to be as follows:

$$\begin{aligned}
\boldsymbol{\sigma}_k^{(2n)} &= (-k^2)^n ik\,\{a_n(\overline{\boldsymbol{e}_k \boldsymbol{u}_k} - 2\boldsymbol{g}_k(\boldsymbol{e}_k \cdot \boldsymbol{u}_k)) + b_n \boldsymbol{g}_k(\boldsymbol{e}_k \cdot \boldsymbol{u}_k)\}\,, \\
\boldsymbol{\sigma}_k^{(2n+1)} &= (-k^2)^{n+1} \boldsymbol{g}_k\{c_n T_k + d_n \rho_k\}\,, \\
\boldsymbol{q}_k^{(2n)} &= (-k^2)^n ik\boldsymbol{e}_k\{\gamma_n T_k + \delta_n \rho_k\}\,, \\
\boldsymbol{q}_k^{(2n+1)} &= (-k^2)^{n+1}\{\alpha_n \boldsymbol{e}_k(\boldsymbol{e}_k \cdot \boldsymbol{u}_k) + \beta_n(\boldsymbol{u}_k - \boldsymbol{e}_k(\boldsymbol{e}_k \cdot \boldsymbol{u}_k)\}\,,
\end{aligned} \tag{8.120}$$

where $\boldsymbol{g}_k = 1/2\overline{\boldsymbol{e}_k \boldsymbol{e}_k}$, and the real-valued coefficients $a_n, \ldots, \beta_n$ are due to the Chapman–Enskog procedure (8.7) and (8.8).

The expressions just presented suggest that the dynamic invariant form of the stress tensor and of the heat flux reads:

$$\begin{aligned}
\boldsymbol{\sigma}_k &= ikA(\overline{\boldsymbol{e}_k \boldsymbol{u}_k} - 2\boldsymbol{g}_k(\boldsymbol{e}_k \cdot \boldsymbol{u}_k)) + 2ikB\boldsymbol{g}_k(\boldsymbol{e}_k \cdot \boldsymbol{u}_k) \\
&\quad -2k^2 C\boldsymbol{g}_k T_k - 2k^2 D\boldsymbol{g}_k \rho_k\,, \\
\boldsymbol{q}_k &= ikZ\boldsymbol{e}_k T_k + ikU\boldsymbol{e}_k \rho_k \\
&\quad -k^2 X(\boldsymbol{u}_k - \boldsymbol{e}_k(\boldsymbol{e}_k \cdot \boldsymbol{u}_k)) - k^2 Y\boldsymbol{e}_k(\boldsymbol{e}_k \cdot \boldsymbol{u}_k)\,,
\end{aligned} \tag{8.121}$$

where the functions $A, \ldots, Y$ depend on $\boldsymbol{k}$. The dynamic invariance condition results in the following two closed systems for these functions:

$$\frac{2}{5}U + D - B + \frac{2}{3}k^2CU + \frac{4}{3}k^2BD = 0\,, \quad (8.122)$$
$$\frac{2}{5}Z + C - B + \frac{2}{3}k^2CZ + \frac{4}{3}k^2BC = 0\,,$$
$$-1 + \frac{2}{5}k^2Y - B - \frac{2}{3}k^2C - k^2D - \frac{4}{3}k^2B^2 + \frac{2}{3}k^2k^2CY = 0\,,$$
$$\frac{4}{3}k^2D - \frac{2}{3}U - k^2Y - \frac{2}{3}k^2ZU + \frac{4}{3}k^2k^2YD = 0\,,$$
$$-\frac{5}{2} + \frac{4}{3}k^2C - \frac{2}{3}Z - k^2Y - \frac{2}{3}k^2Z^2 + \frac{4}{3}k^2k^2YC = 0\,,$$
$$\frac{4}{3}B + \frac{2}{3}Y - U - \frac{2}{3}Z + \frac{2}{3}k^2ZY + \frac{4}{3}k^2YB = 0\,,$$

and

$$-1 - A + \frac{2}{5}k^2X - k^2A^2 = 0\,, \quad (8.123)$$
$$A + \frac{2}{3}X + k^2AX = 0$$

The method of summation of the Chapman–Enskog expansion can also be developed, starting with the structure of the Chapman–Enskog coefficients (8.120), in the same manner as in Sect. 8.3. Simple but rather extensive computations in this case lead, of course, to the invariance equations (8.122) and (8.123).

The Newton method, as applied to the systems (8.122) and (8.123) with the initial Euler approximation, leads in the first iteration to the regularization of the Burnett approximation reported earlier in [43].

Introducing the functions $\bar{A} = k^2A$ and $\bar{X} = k^2X$ in (8.123) we obtain:

$$R(\bar{A}) = \frac{5\bar{A}(3\bar{A}^2 + 5\bar{A} + 2)}{4(6\bar{A} + 5)} = -k^2\,, \quad (8.124)$$

while

$$\bar{X} = -\frac{3\bar{A}}{2 + 3\bar{A}}\,.$$

The derivative, $dR(\bar{A})/d\bar{A}$, becomes equal to zero for $\bar{A}_c \approx -0.364$, which gives the critical wave vector $k_c = \sqrt{-R(\bar{A}_c)} \approx 0.305$. The Newton method, as applied to (8.124) with the initial Euler condition $\bar{A} = 0$, gives the following: the results of the first and of the second iterations are regular functions, while the third and the further iterations bring a singularity which converges to the point k_c (see Fig. 8.9). These singularities (the real poles) of the Newton corrections are of the same nature as discussed above.

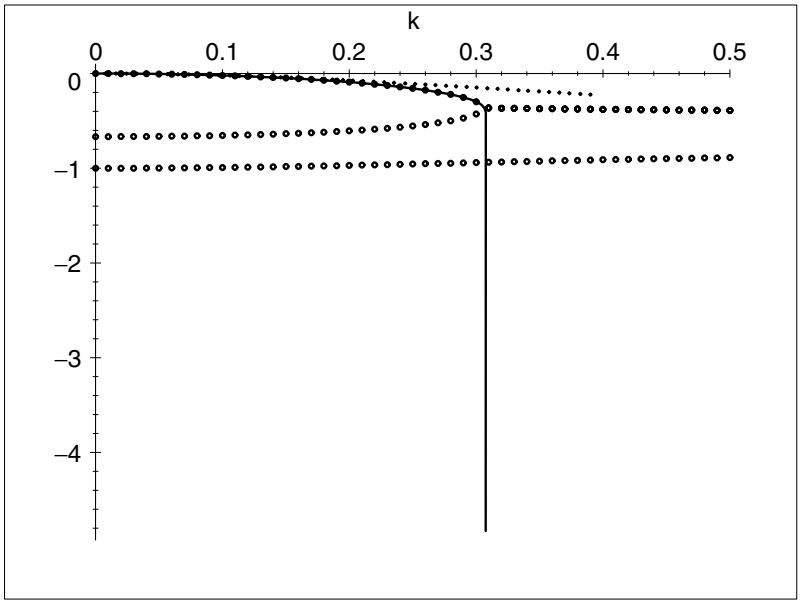

Fig. 8.9. Solutions to (8.123). *Circles*: Numerical solutions. *Dots*: The first Newton iteration. *Solid*: The 4th Newton iteration

8.4.6 Gradient Expansions in Kinetic Theory of Phonons

Exact Chapman–Enskog Solution and Onset of Second Sound

In this section, we close our discussion of linearized Grad systems with an application to simple models for phonon transport in rigid insulators. It is demonstrated that the extended diffusion mode transforms into a second sound mode due to its coupling to a non-hydrodynamic mode at some critical value of the wave vector. This criticality shows up as a branching point of the extension of the diffusion mode within the Chapman–Enskog method. Although the analysis is essentially similar to the examples considered above, it is presented in some details for the sake of completeness.

Experiments on heat pulse propagation through crystalline media [206, 207] confirmed the existence of a temperature window (the Guyer-Krumhansl window [208–210]) with respect to which the features of heat propagation are qualitatively different: At temperatures exceeding the high-temperature edge of the window, the heat propagates in a diffusion-like way. Below the low-temperature edge of the window, the propagation goes in a ballistic way, with the constant speed of sound. Within the window, the propagation becomes wave-like. This latter regime is called second sound (see [211] for a review).

This problem has drawn some renewed attention in the last years. Models relevant for a unified description of diffusion, second sound, and ballistic

regimes of heat propagation are intensively discussed (see [212, 251] and references therein). To be specific, recall the simplest and typical model of the phonon transport [212]. Let $e(\boldsymbol{x}, t)$ and $\boldsymbol{p}(\boldsymbol{x}, t)$ be small deviations of the energy density and energy flux of the phonon field from their equilibrium values, respectively. Then

$$\partial_t e = -c^2 \nabla \cdot \boldsymbol{p} \,, \tag{8.125}$$

$$\partial_t \boldsymbol{p} = -\frac{1}{3} \nabla e - \frac{1}{\tau_R} \boldsymbol{p} \,. \tag{8.126}$$

Here c is the Debye velocity of phonons, and τ_R is the characteristic time of resistive processes. Equations (8.125) can be derived from the Boltzmann–Peierls kinetic equation, within the relaxation time approximation, by a method similar to Grad's method [212]. Equations (8.125), (8.126) provide the simplest model of coupling between the hydrodynamic variable e and the non-hydrodynamic variable $\boldsymbol{p}$, allowing for a qualitative description of both the diffusion and the second sound. Following the standard argumentation [212], we observe the two limiting cases:

1. As $\tau_R \to 0$, equation (8.126) yields the Fourier relation $\boldsymbol{p} = -\frac{1}{3}\tau_R \nabla e$ which closes (8.125) to give the diffusion equation:

$$\partial_t e + \frac{1}{3} \tau_R c^2 \Delta e = 0 \,. \tag{8.127}$$

2. As $\tau_R \to \infty$, (8.126) yields $\partial_t \boldsymbol{p} = -\frac{1}{3}\nabla e$, and (8.125) closes to give the wave equation:

$$\partial_t^2 e + \frac{1}{3} c^2 \Delta e = 0 \,. \tag{8.128}$$

Equation (8.127) describes the usual diffusive regime of heat propagation, while (8.128) is relevant to the (undamped) second sound regime with the velocity $u_2 = c/\sqrt{3}$; both are closed with respect to the variable e.

However, even within the simplest model (8.125), (8.126), the problem of closure remains unsolved in a systematic way when τ_R is finite. The natural way of doing so is provided by the Chapman–Enskog method. In the situation under consideration, the Chapman–Enskog method yields an extension of the diffusive transport to finite values of the parameter τ_R, and leads to an expansion of the non-hydrodynamic variable $\boldsymbol{p}$ in terms of the hydrodynamic variable e. With this, if we are able to make this extension of the diffusive mode exactly, we could learn more about the transition between the diffusion and second sound (within the framework of the model).

The Chapman–Enskog method, as applied to (8.125), (8.126), results in the following series representation:

$$\boldsymbol{p} = \sum_{n=0}^{\infty} \boldsymbol{p}^{(n)} \,, \tag{8.129}$$

where the coefficients $\boldsymbol{p}^{(n)}$ are due to the Chapman–Enskog recurrence procedure,

$$\boldsymbol{p}^{(n)} = -\tau_R \sum_{m=0}^{n-1} \partial_t^{(m)} \boldsymbol{p}^{(n-1-m)} \,, \tag{8.130}$$

while the Chapman–Enskog operators $\partial_t^{(m)}$ act on e as follows:

$$\partial_t^{(m)} e = -c^2 \nabla \cdot \boldsymbol{p}^{(m)} \,. \tag{8.131}$$

Finally, the zero order term reads: $\boldsymbol{p}^{(0)} = -\frac{1}{3}\tau_R \nabla e$, and leads to the Fourier approximation of the energy flux.

To sum up the series (8.129) in closed form, we shall specify the nonlinearity appearing in equations (8.130) and (8.131). The coefficients $\boldsymbol{p}^{(n)}$ in equations (8.129) and (8.130) have the following explicit structure for arbitrary order $n \geq 0$:

$$\boldsymbol{p}^{(n)} = a_n \Delta^n \nabla e \,, \tag{8.132}$$

where the real-valued and yet unknown coefficients a_n are due to the recurrence procedure (8.130), and (8.131). Indeed, the form (8.132) is true for $n = 0$ ($a_0 = -\frac{1}{3}\tau_R$). Let us assume that (8.132) is proven up to order $n-1$. Then, computing the nth order coefficient $\boldsymbol{p}^{(n)}$, we derive:

$$\begin{aligned} \boldsymbol{p}^{(n)} &= -\tau_R \sum_{m=0}^{n-1} \partial_t^{(m)} a_{n-1-m} \Delta^{(n-1-m)} \nabla e \\ &= -\tau_R \sum_{m=0}^{n-1} a_{n-1-m} \Delta^{(n-1-m)} \nabla \left(-c^2 a_m \nabla \cdot \nabla \Delta^m e \right) \\ &= \tau_R c^2 \left\{ \sum_{m=0}^{n-1} a_{n-1-m} a_m \right\} \Delta^n \nabla e \,. \end{aligned} \tag{8.133}$$

The last expression has the same form as (8.132). Thus, the Chapman–Enskog procedure for the model (8.125), (8.126) is equivalent to the following nonlinear recurrence relation in terms of the coefficients a_n:

$$a_n = \tau_R c^2 \sum_{m=0}^{n-1} a_{n-1-m} a_m \,, \tag{8.134}$$

subject to the initial condition $a_0 = -\frac{1}{3}\tau_R$. Further, it is convenient to make the Fourier transform. Using $\boldsymbol{p} = \boldsymbol{p}_k \exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}$ and $e = e_k \exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}$, where $\boldsymbol{k}$ is the real-valued wave vector, we derive in (8.132): $\boldsymbol{p}_k^{(n)} = a_n i\boldsymbol{k}(-k^2)^n e_k$, and

$$\boldsymbol{p}_k = i\boldsymbol{k} A(k^2) e_k \,, \tag{8.135}$$

where

$$A(k^2) = \sum_{n=0}^{\infty} a_n (-k^2)^n \ . \tag{8.136}$$

Thus, the the Chapman–Enskog solution (8.129) amounts to finding the function $A(k^2)$ represented by the power series (8.136). If the function A is known, the exact Chapman–Enskog closure of the system (8.125), (8.126) amounts to the following dispersion relation of plane waves $\sim \exp\{\omega_k t + i\boldsymbol{k}\cdot\boldsymbol{x}\}$:

$$\omega_k = c^2 k^2 A(k^2) \ . \tag{8.137}$$

Here ω_k is a complex-valued function of the real-valued vector $\boldsymbol{k}$: $\mathrm{Re}(\omega_k)$ is the attenuation rate, $\mathrm{Im}(\omega_k)$ is the frequency.

Multiplying both equations in (8.134) with $(-k^2)^n$, and performing a summation in n from 1 to infinity, we get:

$$A - a_0 = -\tau_R c^2 k^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n-m} (-k^2)^{n-m} a_m (-k^2)^m \ ,$$

Now we notice that

$$\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{n} a_{n-m} (-k^2)^{n-m} a_m (-k^2)^m = A^2 \ ,$$

Setting $a_0 = -\frac{1}{3}\tau_R$, we derive a quadratic equation for the function A:

$$\tau_R c^2 k^2 A^2 + A + \frac{1}{3}\tau_R = 0 \ . \tag{8.138}$$

Further, a selection procedure is required to choose the relevant root of (8.138). Firstly, recall that all the coefficients a_n (8.132) are real-valued by the sense of the Chapman–Enskog method (8.130) and (8.131), hence the function A (8.136) is real-valued. Therefore, only the real-valued root of (8.138) is relevant to the Chapman–Enskog solution. The first observation is that (8.138) has no real-valued solutions as soon as k becomes bigger than the critical value k_c, where

$$k_\mathrm{c} = \frac{\sqrt{3}}{2\tau_R c} \ . \tag{8.139}$$

Secondly, there are two real-valued solutions to (8.138) at $k < k_\mathrm{c}$. However, only one of them satisfies the Chapman–Enskog asymptotic $\lim_{k\to 0} A(k^2) = -\frac{1}{3}\tau_R$.

With these two remarks, we finally derive the following exact Chapman–Enskog dispersion relation (8.137):

$$\omega_k = \begin{cases} -(2\tau_R)^{-1} \left(1 - \sqrt{1 - (k^2)/(k_\mathrm{c}^2)}\right) & k < k_\mathrm{c} \\ \text{none} & k > k_\mathrm{c} \end{cases} \ . \tag{8.140}$$

This dispersion relation corresponds to the extended diffusion transport, and it comes back to the standard Fourier approximation in the limit of long waves $k/k_c \ll 1$. The Chapman–Enskog solution does not exist as soon as $k/k_c > 1$. For $k = k_c$, the extended diffusion branch crosses one of the non-hydrodynamic branches of (8.125), (8.126). For larger k, the extended diffusion mode and the critical non-hydrodynamic mode produce a pair of complex conjugate solutions with the real part equal to $-\frac{1}{2\tau_R}$. The imaginary part of this extension after k_c attains the asymptotic value $\pm iu_2 k$, as $k \to \infty$, where $u_2 = c/\sqrt{3}$ is the (undamped) second sound velocity in the model (8.125), (8.126) (see equation (8.128)). Although the spectrum of the original (8.125), (8.126) continues indeed after k_c, the Chapman–Enskog method does not recognize this extension as part of the hydrodynamic branch, *while the second sound regime is born from the extended diffusion after coupling with the critical non-hydrodynamic mode.*

Finally, let us consider the opportunities provided by the Newton method as applied to the invariance equation. First, the invariance equation can be easily obtained in closed form here. Consider again the expression for the heat flux in terms of the energy density (8.135), $\boldsymbol{p}_k = i\boldsymbol{k}A(k^2)e_k$, where now the function A is not thought as the Chapman–Enskog series (8.136). The invariance equation is a constraint on the function A, expressing the form-invariance of the heat flux (8.135) under both the dynamic equations (8.125) and (8.126). Computing the time derivative of function (8.135) due to equation (8.125), we obtain:

$$\partial_t^{\text{macro}} \boldsymbol{p}_k = i\boldsymbol{k}A(k^2)\partial_t e_k = c^2 k^2 A^2 i\boldsymbol{k}e_k \,. \tag{8.141}$$

On the other hand, computing the time derivative of the same function due to equation (8.126), we have:

$$\partial_t^{\text{micro}} \boldsymbol{p}_k = -\frac{1}{3} i\boldsymbol{k}e_k - \frac{1}{\tau_R} A i\boldsymbol{k}e_k \,. \tag{8.142}$$

Equating (8.141) and (8.142), we derive the desired invariance equation for the function A. This equation coincides with the exact Chapman–Enskog equation (8.138).

As the second step, let us apply the Newton method to the invariance equation (8.138), taking the Euler approximation ($A_0^N \equiv 0$) as the initial condition. Rewriting (8.138) in the form $F(A, k^2) = 0$, we come to the following Newton iterations:

$$\left.\frac{\mathrm{d}F(A,k^2)}{\mathrm{d}A}\right|_{A=A_n} (A_{n+1} - A_n) + F(A_n, k^2) = 0 \,. \tag{8.143}$$

The first two iterations give:

$$\tau_R^{-1} A_1 = -\frac{1}{3} \,, \tag{8.144}$$

$$\tau_R^{-1} A_2 = -\frac{1 - \frac{1}{4}y^2}{3(1 - \frac{1}{2}y^2)} \,. \tag{8.145}$$

The first Newton iteration (8.144) coincides with the first term of the Chapman–Enskog expansion. The second Newton iteration (8.145) is a rational function with the Taylor expansion coinciding with the Chapman–Enskog solution up to the super-Burnett term, and it has a pole at $y_2 = \sqrt{2}$. The further Newton iterations are also rational functions with the relevant poles at points y_n, and the sequence of this points tends very rapidly to the location of the actual singularity $y_c = 1$ ($y_3 \approx 1.17$, $y_4 \approx 1.01$, etc.).

Inclusion of Normal Processes

Accounting for normal processes in the framework of the semi-hydrodynamical models [212] leads to the following generalization of (8.125), (8.126) (written in Fourier variables for the one-dimensional case):

$$\partial_t e_k = -ikc^2 p_k \,, \tag{8.146}$$

$$\partial_t p_k = -\frac{1}{3} ike_k - ikN_k - \frac{1}{\tau_R} p_k \,, \tag{8.147}$$

$$\partial_t N_k = -\frac{4}{15} ikc^2 p_k - \frac{1}{\tau} N_k \,, \tag{8.148}$$

where $\tau = \tau_N \tau_R/(\tau_N + \tau_R)$, τ_N is the characteristic time of normal processes, and N_k is the additional field variable. Following the principle of invariance as explained in the preceding section, we write the closure relation for the non-hydrodynamic variables p_k and N_k as:

$$p_k = ikA_k e_k, \quad N_k = B_k e_k \,, \tag{8.149}$$

where A_k and B_k are two unknown functions of the wave vector k. Further, following the principle of invariance as explained in the previous section, each of the relations (8.149) should be invariant under the dynamics due to (8.146), and due to (8.147) and (8.148). This results in two equations for the functions A_k and B_k:

$$k^2 c^2 A_k^2 = -\frac{1}{\tau_R} A_k - B_k - \frac{1}{3} \,, \tag{8.150}$$

$$k^2 c^2 A_k B_k = -\frac{1}{\tau} B_k + \frac{4}{15} k^2 c^2 A_k \,.$$

When the energy balance equation (8.146) is closed with the relation (8.149), this amounts to a dispersion relation for the extended diffusion mode, $\omega_k = k^2 c^2 A_k$, where A_k is the solution to the invariance equations (8.150), *subject to the condition* $A_k \to 0$ as $k \to 0$. Resolving equations (8.150) with respect to A_k, and introducing $\overline{A}_k = k^2 c^2 A_k$, we arrive at the following:

$$\Phi(\overline{A}_k) = \frac{5\overline{A}_k(1 + \tau\overline{A}_k)(\tau_R \overline{A}_k + 1)}{5 + 9\tau\overline{A}_k} = -\frac{1}{3}\tau_R k^2 c^2 \,. \tag{8.151}$$

The invariance equation (8.151) is completely analogous to the (8.138). Written in the form (8.151), it allows for a direct investigation of the critical points. For this purpose, we find zeroes of the derivative, $d\Phi(\overline{A}_k)/d\overline{A}_k = 0$. When the roots of the latter equation, $\overline{A}_k^c$, are found, the critical values of the wave vector are given as $-(1/3)k_c^2c^2 = \Phi(\overline{A}_k^c)$. The condition $d\Phi(\overline{A}_k)/d\overline{A}_k = 0$ reads:

$$18\tau^2\tau_R\overline{A}_k^3 + 3\tau(3\tau + 8\tau_R)\overline{A}_k^2 + 10(\tau + \tau_R)\overline{A}_k + 5 = 0\ . \tag{8.152}$$

Let us consider the particularly interesting case, $\epsilon = \tau_N/\tau_R \ll 1$ (the normal events are less frequent than resistive). Then the real-valued root of (8.152), $\overline{A}_k(\epsilon)$, corresponds to the coupling of the extended diffusion mode to the critical non-hydrodynamic mode. The corresponding modification of the critical wave vector k_c (8.139) due to the normal processes amounts to shifts towards shorter waves, and we derive:

$$[k_c(\epsilon)]^2 = k_c^2 + \frac{3\epsilon}{10\tau_R^2c^2}\ . \tag{8.153}$$

Accounting for Anisotropy

The above examples concerned the isotropic Debye model. Let us consider the simplest anisotropic model of a cubic media with a longitudinal (L) and two degenerated transverse (T) phonon modes, taking into account resistive processes only. Introduce the Fourier variables, e_k, e_k^T, $\boldsymbol{p}_k^T$, and $\boldsymbol{p}_k^L$, where $e_k = e_k^L + 2e_k^T$ is the Fourier transform of the total energy of the three phonon modes (the only conserved quantity), while the rest of variables are specific quantities. The isotropic model (8.125), (8.126) generalizes to [212]:

$$\partial_t e_k = -ic_L^2\boldsymbol{k}\cdot\boldsymbol{p}_k^L - 2ic_T^2\boldsymbol{k}\cdot\boldsymbol{p}_k^T\ , \tag{8.154}$$

$$\partial_t e_k^T = -ic_T^2\boldsymbol{k}\cdot\boldsymbol{p}_k^T + \frac{1}{\lambda}\left[c_L^3(e_k - 2e_k^T) - c_T^3e_k^T\right]\ , \tag{8.155}$$

$$\partial_t \boldsymbol{p}_k^L = -\frac{1}{3}i\boldsymbol{k}(e_k - 2e_k^T) - \frac{1}{\tau_R^L}\boldsymbol{p}_k^L\ , \tag{8.156}$$

$$\partial_t \boldsymbol{p}_k^T = -\frac{1}{3}i\boldsymbol{k}e_k^T - \frac{1}{\tau_R^T}\boldsymbol{p}_k^T\ , \tag{8.157}$$

where $\lambda = \tau_R^Tc_T^3 + 2\tau_R^Lc_L^3$. The term containing the factor λ^{-1} corresponds to the energy exchange between the L and T phonon modes. The invariance constraint for the closure relations,

$$\boldsymbol{p}_k^L = i\boldsymbol{k}A_ke_k\ ,\quad \boldsymbol{p}_k^T = i\boldsymbol{k}B_ke_k\ ,\quad e_k^T = X_ke_k\ , \tag{8.158}$$

result in the following invariance equations for the $\boldsymbol{k}$-dependent functions A_k, B_k, and X_k:

$$k^2 c_L^2 A_k^2 + 2k^2 c_T^2 A_k B_k = -\frac{1}{\tau_R^L} A_k - \frac{1}{3}\left(1 - 2X_k\right) \ , \tag{8.159}$$

$$2k^2 c_T^2 B_k^2 + k^2 c_L B_k A_k = -\frac{1}{\tau_R^T} B_k - \frac{1}{3} X_k \ , \tag{8.160}$$

$$X_k\left(k^2 c_L^2 A_k + 2k^2 c_T^2 B_k\right) = c_T^2 k^2 B_k + \frac{1}{\lambda}\left[c_L^3 - X_k\left(2c_L^3 + c_T^3\right)\right] \ . \tag{8.161}$$

When the energy balance equation (8.154) is closed with the relations (8.158), we obtain the dispersion relation for the extended diffusion mode, $\omega_k = \overline{A}_k + 2\overline{B}_k$, where the functions $\overline{A}_k = k^2 c_L^2 A_k$, and $\overline{B}_k = k^2 c_T^2 B_k$, satisfy the conditions: $\overline{A}_k \to 0$, and $\overline{B}_k \to 0$, as $k \to 0$. The resulting dispersion relation is rather complicated in the general case of the four parameters of the problem, c_L, c_T, τ_R^L and τ_R^T. Therefore, introducing the function $\overline{Y}_k = \overline{A}_k + 2\overline{B}_k$, let us consider the following specific situations of closed equations for the $\overline{Y}_k$ on the basis of the invariance equations (8.159):

(i) $c_L = c_T = c$, $\tau_R^L = \tau_R^T = \tau_R$ (complete degeneration of the parameters of the L and T subsystems): The system (8.159) results in two decoupled equations:

$$\overline{Y}_k\left(\tau_R \overline{Y}_k + 1\right) + \frac{1}{3} k^2 c^2 \tau_R = 0 \ , \tag{8.162}$$

$$\left(\tau_R \overline{Y}_k + 1\right)^2 + \frac{1}{3} k^2 c^2 \tau_R^2 = 0 \ . \tag{8.163}$$

Equation (8.162) coincides with (8.138) for the isotropic case, and its solution defines the coupling of the extended diffusion to a non-hydrodynamic mode. Equation (8.163) does not have a solution with the required asymptotic behavior $\overline{Y}_k \to 0$ as $k \to 0$, and is therefore irrelevant to the features of the diffusion mode in this completely degenerated case. It describes the two further propagating and damped non-hydrodynamic modes of the (8.154). The nature of these modes, as well of the mode which couples to the diffusion mode well be seen below.

(ii) $c_L = c_T = c$, $\tau_R^L \neq \tau_R^T$ (nondegenerate characteristic time of resistive processes in the L and T subsystems):

$$\left[\overline{Y}_k\left(\tau_R^L \overline{Y}_k + 1\right) + \frac{1}{3} k^2 c^2 \tau_R^L\right] \times \left[\left(\tau_R' \overline{Y}_k + 3\right)\left(\tau_R^T \overline{Y}_k + 1\right) + \frac{1}{3} k^2 c^2 \tau_R^T \tau_R'\right]$$
$$+ \frac{2}{3} k^2 c^2 \left(\tau_R^T - \tau_R^L\right) = 0 \ , \tag{8.164}$$

where $\tau_R' = 2\tau_R^L + \tau_R^T$. As $\tau_R^T - \tau_R^L \to 0$, (8.164) tends to the degenerated case (8.162). At $k = 0$, $\tau_R^L \neq \tau_R^L$, there are four solutions to (8.164). The $\overline{Y}_0 = 0$ is the hydrodynamic solution indicating the beginning of the diffusion mode. The two non-hydrodynamic solutions, $\overline{Y}_0 = -1/\tau_R^L$, and $\overline{Y}_0 = -1/\tau_R^T$, $\overline{Y}_0 = -3/\tau_R'$, are associated with the longitudinal and the transverse phonons, respectively. The difference in relaxational times makes

the latter transverse root nondegenerate, and the third non-hydrodynamic mode, $\overline{Y}_0 = -3/\tau_R'$, appears instead.

(iii) $c_L \neq c_T$, $\tau_R^L = \tau_R^T = \tau_R$ (nondegenerate speed of the L and the T sound).

$$\left[\overline{Y}_k\left(\tau_R\overline{Y}_k + 1\right) + \frac{1}{3}k^2c_L^2\tau_R\right] \times \left[\left(\tau_R\overline{Y}_k + 1\right)^2 + \frac{1}{3}k^2c_T^2\tau_R^2\right]$$
$$+\frac{2}{3}k^2\tau_R\frac{c_L^3(c_T^2 - c_L^2)}{2c_L^3 + c_T^3}\left(\tau_R\overline{Y}_k + 1\right) = 0\,. \tag{8.165}$$

As $c_T - c_L \to 0$, (8.165) tends to the degenerate case (8.162). However, this time the non-hydrodynamic mode associated with the transverse phonons degenerates at $k = 0$.

Thus, we are able to identify the modes in (8.162) and (8.163). The non-hydrodynamic mode which couples to the extended diffusion mode is associated with the longitudinal phonons, and is the case (8.162). The case (8.163) is due to the transverse phonons. In the nondegenerate cases, (8.164) and (8.165), both pairs of modes become propagating after a certain critical values of k, and the behavior of the extended diffusion mode is influenced by all three non-hydrodynamic modes just mentioned. It should be stressed, however, that the second sound mode, which is the continuation of the diffusion mode [206, 207], is due to (8.162). The results of the above analysis lead to the following conclusion:

(i) The examples considered above indicate an interesting mechanism of a *kinetic* formation of the second sound regime from the extended diffusion with the participation of the non-hydrodynamic mode. The onset of the propagating mode shows up as the critical point of the extension of the hydrodynamic solution into the domain of finite k, which was found within the Chapman–Enskog and equivalent approaches. These results concern the situation at the high-temperature edge of the Guyer–Krumhansl window, and are complementary to the coupling between the transversal ballistic mode and the second sound at the low-temperature edge [213].

(ii) The crossover from the diffusion-like to the wave-like propagation was previously found in [214–216] in the framework of exact Chapman–Enskog solution to the Boltzmann equation for the Lorentz gas model [202], and for similar models of phonon scattering in anisotropic disordered media [217]. The characteristic common feature of the models studied in [202, 214–217] and the models [212] is the existence of a gap between the hydrodynamic (diffusive) and the non-hydrodynamic components of the spectrum. Therefore, one can expect that the destruction of the extended diffusion is solely due to the *existence* of this gap. In applications to the phonon kinetic theory this amounts to the introduction of the relaxation time approximation. In other words, we may expect that the mechanism of crossover from diffusion to second sound in the simple models [212] is identical to what could be found from the phonon-Boltzmann kinetic equation

within the relaxation time approximation. However, it should be noted that the original (i.e., without the relaxation time approximation) phonon kinetic equations are *gapless* (see, for example, [211]). On the other hand, most of the works on heat propagation in solids *do* exploit the idea of the gap, since it is only possible to speak of diffusion if such a gap exists. To conclude this point, the following general hypothesis can be expressed: *the existence of diffusion (and hence of the gap in the relaxational spectrum) leads to its destruction through the coupling with a non-hydrodynamic mode.*

8.4.7 Nonlinear Grad Equations

In the preceding sections, the Chapman–Enskog and other methods were probed explicitly for the linearized Grad equations far beyond the usual Navier–Stokes approximation. This was possible, first of all because the problem of the reduced description was shaped into a rather simple *algebraic* form. Indeed, the algebraic structure of the stress tensor $\boldsymbol{\sigma}_k(\rho_k, \boldsymbol{u}_k, T_k, \boldsymbol{k})$ and of the heat flux $\boldsymbol{q}_k(\rho_k, \boldsymbol{u}_k, T_k, \boldsymbol{k})$ was fairly simple. However, when we attempt to extend the approach onto the nonlinear Grad equations, the algebraic structure of the problem is no longer simple. Indeed, when we proceed along the lines of the Chapman–Enskog method, for example, the number of *types* of terms, $\nabla\boldsymbol{u}$, $\nabla\nabla\boldsymbol{u}$, $(\nabla\boldsymbol{u})^2$, $\nabla T\nabla\rho$, and so on, in the Chapman–Enskog coefficients $\boldsymbol{\sigma}^{(n)}$ and $\boldsymbol{q}^{(n)}$ demonstrates a combinatorial growth with the order n.

Still, progress is possible if we impose some rules for the selection of the relevant terms. As applied to the Chapman–Enskog expansion, these selection rules prescribe that only contributions arising from terms with a definite structure in each order $\boldsymbol{\sigma}^{(n)}$ and $\boldsymbol{q}^{(n)}$ should be retained, and all other terms should be ignored. This approach can be linked again with the partial summation rules for the perturbation series in many-body theories, where usually terms with a definite structure are summed instead of the whole series. Our viewpoint on the problem of the extension of the hydrodynamics in the nonlinear case can be expressed as follows: The exact extension seems to be impossible, and, moreover, quite useless because of the lack of a physical transparency. Instead, certain sub-series of the Chapman–Enskog expansion, selected on clear physical grounds, may lead to less complicated equations, which, at the same time, provide an extension for a certain subclass of hydrodynamic phenomena. This viewpoint is illustrated in this section by considering a sub-series of the Chapman–Enskog expansion which provides the dominating contribution when the flow velocity becomes very large (and thus it is relevant to a high-speed subclass of hydrodynamic phenomena such as strong shock waves).

The approach to the Chapman–Enskog series for the nonlinear Grad equations just mentioned, and which was based on a diagrammatic representation of the Chapman–Enskog method, has been attempted earlier in [237]. In this section, however, we shall take the route of the dynamic invariance equations which leads to the same results more directly.

The Dynamic Viscosity Factor

The starting point is the set of one-dimensional nonlinear Grad equations for the hydrodynamic variables ρ, u and T, coupled to the non-hydrodynamic variable σ, where σ is the xx-component of the stress tensor:

$$\partial_t \rho = -\partial_x(\rho u) \; ; \tag{8.166}$$

$$\partial_t u = -u\partial_x u - \rho^{-1}\partial_x p - \rho^{-1}\partial_x \sigma \; ; \tag{8.167}$$

$$\partial_t T = -u\partial_x T - (2/3)T\partial_x u - (2/3)\rho^{-1}\sigma\partial_x u \; ; \tag{8.168}$$

$$\partial_t \sigma = -u\partial_x \sigma - (4/3)p\partial_x u - (7/3)\sigma\partial_x u - \frac{p}{\mu(T)}\sigma \; . \tag{8.169}$$

Here $\mu(T)$ is the temperature-dependent viscosity coefficient. We shall adopt the form $\mu(T) = \alpha T^{\gamma}$, which is characteristic to the point-center models of particles collisions, where γ varies from $\gamma = 1$ (the Maxwell molecules) to $\gamma = 1/2$ (hard spheres), and where α is a dimensional factor.

Even in this model, the Chapman–Enskog expansion appears to be exceedingly complicated in the full setting. Therefore, we address another, simpler problem: *What is the leading correction to the Navier–Stokes approximation when the characteristic value of the average velocity is comparable to the thermal velocity?*

Our goal is to compute the correction to the Navier–Stokes closure relation, $\sigma_{\rm NS} = -(4/3)\mu\partial_x u$, for high values of the average velocity. Let us consider first the Burnett correction from (8.166):

$$\sigma_{\rm B} = -\frac{4}{3}\mu\partial_x u + \frac{8(2-\gamma)}{9}\mu^2 p^{-1}(\partial_x u)^2 - \frac{4}{3}\mu^2 p^{-1}\partial_x(\rho^{-1}\partial_x p) \; . \tag{8.170}$$

The correction of the desired type is given by the nonlinear term proportional to $(\partial_x u)^2$. Each further nth term of the Chapman–Enskog expansion contributes, among others, a nonlinear term proportional to $(\partial_x u)^{n+1}$. Such terms can be named *high-speed* terms since they dominate the rest of the contributions in each order of the Chapman–Enskog expansion when the characteristic average velocity is comparable to the heat velocity. Indeed, if U is a characteristic mean velocity, and $u = U\overline{u}$, where $\overline{u}$ is dimensionless, then the term $(\partial_x u)^{n+1}$ receives the factor U^{n+1} which is the highest possible order of U among the terms available in the nth order of the Chapman–Enskog expansion. Simple dimensional analysis leads to the conclusion that such terms are of the form $\mu g^n \partial_x u$, where $g = p^{-1}\mu\partial_x u$ is dimensionless. Therefore, the Chapman–Enskog expansion for the function σ may be formally rewritten as:

$$\sigma = -\mu\left\{\frac{4}{3} - \frac{8(2-\gamma)}{9}g + r_2 g^2 + \ldots + r_n g^n + \ldots\right\}\partial_x u + \ldots \tag{8.171}$$

The series in the brackets is the collection of the high-speed contributions of interest, coming from *all* orders of the Chapman–Enskog expansion, while

the dots outside the brackets stand for the terms of other nature. Thus, the high-speed correction to the Navier–Stokes closure relation in the framework of the Grad equations (8.166) takes the form:

$$\sigma_{\rm nl} = -\mu R(g)\partial_x u \ , \tag{8.172}$$

where $R(g)$ is the function represented by a formal subsequence of Chapman–Enskog terms in the expansion (8.171). The function R can be viewed also as a dynamic modification of the viscosity μ due to the gradient of the average velocity.

We shall now turn to the problem of an explicit derivation of the function R (8.172). Following the principle of dynamic invariance, we first compute the microscopic derivative of the function $\sigma_{\rm nl}$ by substituting (8.172) into the right hand side of (8.169):

$$\begin{aligned}\partial_t^{\rm micro}\sigma_{\rm nl} &= -u\partial_x\sigma_{\rm nl} - \frac{4}{3}p\partial_x u - \frac{7}{3}\sigma_{\rm nl}\partial_x u - \frac{p}{\mu(T)}\sigma_{\rm nl} \\ &= \left\{-\frac{4}{3} + \frac{7}{3}gR + R\right\} p\partial_x u + \dots \ ,\end{aligned} \tag{8.173}$$

where dots denote the terms irrelevant to the closure relation (8.172) (such terms appear, because (8.172) is not the exact closure relation).

Second, computing the macroscopic derivative of the closure relation (8.172) due to (8.166), (8.167), and (8.168), we obtain:

$$\partial_t^{\rm macro}\sigma_{\rm nl} = -[\partial_t\mu(T)]R\partial_x u - \mu(T)\frac{{\rm d}R}{{\rm d}g}[\partial_t g]\partial_x u - \mu(T)R\partial_x[\partial_t u] \ . \tag{8.174}$$

In the latter expression, the time derivatives of the hydrodynamic variables should be replaced with the right hand sides of (8.166), (8.167), and (8.168), where, in turn, the function σ should be replaced by the function $\sigma_{\rm nl}$ (8.172). After some computation, we derive the following:

$$\partial_t^{\rm macro}\sigma_{\rm nl} = \left\{gR + \frac{2}{3}(1-gR)\times\left(\gamma gR + (\gamma-1)g^2\frac{{\rm d}R}{{\rm d}g}\right)\right\} p\partial_x u + \dots \tag{8.175}$$

Again, the dots stand for the terms irrelevant to the present analysis.

Equating the relevant terms in (8.173) and (8.175), we obtain the invariance equation for the function R:

$$(1-\gamma)g^2\,(1-gR)\,\frac{{\rm d}R}{{\rm d}g} + \gamma g^2R^2 + \left[\frac{3}{2} + g(2-\gamma)\right]R - 2 = 0 \ . \tag{8.176}$$

For Maxwell molecules ($\gamma = 1$), (8.176) simplifies considerably, and becomes algebraic:

$$g^2R^2 + \left(\frac{3}{2} + g\right)R - 2 = 0 \ . \tag{8.177}$$

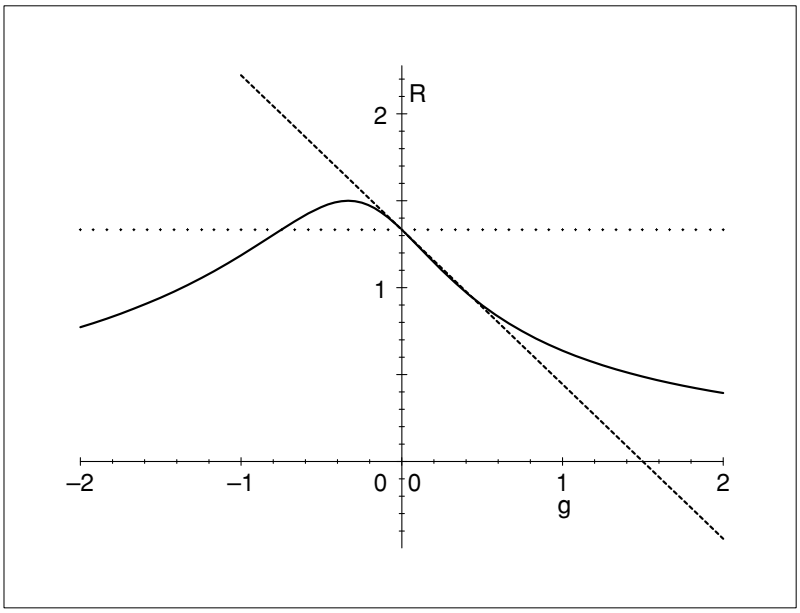

Fig. 8.10. Viscosity factor $R(g)$ for Maxwell molecules. *Solid*: exact solution. *Dash*: the Burnett approximation. *Dots*: the Navier–Stokes approximation

The solution which recovers the Navier–Stokes closure relation in the limit of small g then reads:

$$R_{\mathrm{MM}} = \frac{-3 - 2g + 3\sqrt{1 + (4/3)g + 4g^2}}{4g^2} \,. \tag{8.178}$$

Function R_{MM} (8.178) is plotted in Fig. 8.10. Note that R_{MM} is positive for all values of its argument g, as is appropriate for the viscosity factor, while the Burnett approximation to the function R_{MM} violates positivity.

For other models ($\gamma \neq 1$), the invariance equation (8.176) is a rather complicated nonlinear ODE with the initial condition $R(0) = 4/3$ (the Navier–Stokes condition). Several ways to derive analytic results are possible. One possibility is to expand the function R into powers of g, in the point $g = 0$. This will bring us back to the original sub-series of the Chapman–Enskog expansion (see (8.171)). Instead, we take advantage of the opportunity offered by the parameter γ. Introduce another parameter $\beta = 1 - \gamma$, and consider the expansion:

$$R(\beta, g) = R_0(g) + \beta R_1(g) + \beta^2 R_2(g) + \dots \,.$$

Substituting this expansion into the invariance equation (8.176), we derive $R_0(g) = R_{\mathrm{MM}}(g)$,

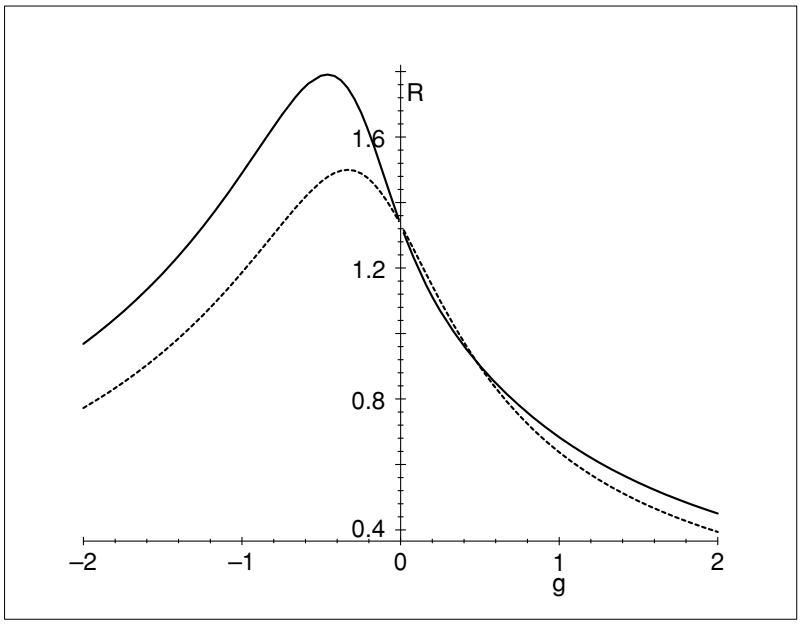

Fig. 8.11. Viscosity factor $R(g)$ for hard spheres. *Solid*: the first approximation. *Dash*: exact solution for the Maxwell molecules

$$R_1(g) = -g(1 - gR_0)\frac{R_0 + g(\mathrm{d}R_0/\mathrm{d}g)}{2g^2R_0 + g + (3/2)}\,, \tag{8.179}$$

etc. That is, the solution for other models is constructed in a form of a series with the exact solution for the Maxwell molecules as the leading term. For hard spheres ($\beta = 1/2$), the result to the first-order term reads: $R_{\mathrm{HS}} \approx R_{\mathrm{MM}} + (1/2)R_1$. The resulting approximate viscosity factor is shown in Fig. 8.11. The features of the approximation obtained are qualitatively the same as in the case of Maxwell molecules.

It is interesting that precisely the same result for the nonlinear elongational viscosity obtained in [17] was derived later by A. Santos [18] from the solution to the BGK kinetic equation in the regime of so-called homoenergetic extension flow, see also [19], where a misprint in [17] in formula (8.179) was detected. For further discussion, see [20]

Attraction to the Invariant Set

Above, we have derived a correction to the Navier–Stokes expression for the stress σ, in the one-dimensional case, for large values of the average velocity u. This correction has the form $\sigma = -\mu R(g)\partial_x u$, where $g \propto \partial_x u$ is the longitudinal rate. The viscosity factor $R(g)$ is a solution to the differential equation (8.176), subject to a certain initial condition. Uribe and Piña [218]

have indicated some interesting features of the invariance equation (8.176) for the model of hard spheres. In particular, they have found that a numerical integration from the initial point into the domain of negative longitudinal rates is very difficult. What happens to the relevant solution for negative values of g?

Let us denote as $P = (g, R)$ the points in the (g, R) plane. The relevant solution $R(g)$ emerges from the point $P_0 = (0, 4/3)$, and can be uniquely continued to arbitrary values of g, positive and negative. This solution can be constructed, for example, with the Taylor expansion, and which is identical with the relevant sub-series of the Chapman–Enskog expansion. However, the difficulty in constructing this solution numerically for $g < 0$ originates from the fact that the same point P_0 is the point of *essential singularity* of other (irrelevant) solutions to (8.176). Indeed, for $|g| \ll 1$, let us consider $\tilde{R}(g) = R(g) + \Delta$, where $R(g) = (4/3) + (8/9)(\gamma - 2)g$ is the relevant solution for small $|g|$, and $\Delta(g)$ is a deviation. Neglecting in (8.176) all regular terms (of the order g^2), and also neglecting $g\Delta$ in comparison to Δ, we derive the following equation: $(1 - \gamma)g^2(\mathrm{d}\Delta/\mathrm{d}g) = -(3/2)\Delta$. The solution is $\Delta(g) = \Delta(g_0)\exp[a(g^{-1} - g_0^{-1})]$, where $a = (3/2)(1-\gamma)^{-1}$. The essential singularity at $g = 0$ is apparent from this solution, unless $\Delta(g_0) \neq 0$ (that is, no singularity exists only for the relevant solution, $\tilde{R} = R$). Let $\Delta(g_0) \neq 0$. If $g < 0$, then $\Delta \to 0$, together with all its derivatives, as $g \to 0$. If $g > 0$, the solution blows up, as $g \to 0$.

The complete picture for $\gamma \neq 1$ is as follows: The lines $g = 0$ and $P = (g, g^{-1})$ define the boundaries of the basin of attraction $A = A_- \bigcup A_+$, where $A_- = \{P| - \infty < g < 0, R > g^{-1}\}$, and $A_+ = \{P|\infty > g > 0, R < g^{-1}\}$. The graph $G = (g, R(g))$ of the relevant solution belongs to the closure of A, and goes through the points $P_0 = (0, 4/3)$, $P_- = (-\infty, 0)$, and $P_+ = (\infty, 0)$. These points at the boundaries of A are the points of essential singularity of any other (irrelevant) solution with the initial condition $P \in A$, $P \notin A \bigcap G$. Namely, if $P \in A_+$, $P \notin A_+ \bigcap G$, the solution blows up at P_0, and attracts to P_+. If $P \in A_-$, $P \notin A_- \bigcap G$, the solution blows up at P_-, and attracts to P_0.

The above consideration is supported by a numerical study of (8.176). In Fig. 8.12, it is demonstrated how the dynamic viscosity factor $R(g)$ emerges as the attractor of various solutions to the invariance equation (8.176) [the case considered corresponds to hard spheres, $\gamma = 1/2$]. The analytical approximation (8.179) is also shown in Fig. 8.12. It provides a reasonable global approximation to the attractor for both positive and negative g. We conclude with a discussion.

(i) The main feature of the above example of extending the hydrodynamic description into a highly non-equilibrium and nonlinear domain can be expressed as follows: this is an *exact partial summation* of the Chapman–Enskog expansion. "Partial" means that the relevant high-speed terms, dominating the other contributions in the limit of the high average velocity, were

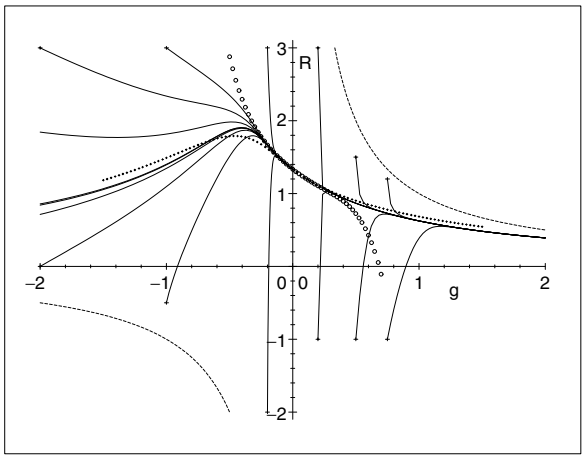

Fig. 8.12. Viscosity factor as an attractor. *Solid* lines: numerical integration with various initial points (crosses). Two poorly resolved lines correspond to the initial conditions $(-100, 0)$ and $(-100, 3)$. *Circles*: Taylor expansion to the 5th order. *Dots*: the analytical approximation of (8.179). *Dash*: boundaries of the basin of attraction

accounted to all orders of the original Chapman–Enskog expansion. "Exact" means that, though we have used the formally different route, the result is indeed the sum of the relevant sub-series of the original Chapman–Enskog expansion. In other words, if we now expand the function $R_{\mathrm{MM}}(g)$ (8.178) in powers of g, around the point $g = 0$, we obtain again to the corresponding series inside the brackets in (8.171). That this is indeed true can be checked up to the few lower orders straightforwardly, although the complete proof requires a more involved analysis. As the final comment to this point, we would like to stress a certain similarity between the problem considered above and the frequent situations in many-body problems: there is no single leading *term*; instead, there is the leading *sub-series* of the perturbation expansions, under certain conditions.

(ii) Let us discuss briefly the features of the resulting hydrodynamics. The hydrodynamic equations are now given by (8.166), (8.167), and (8.168), where σ is replaced by σ_{nl} (8.172). First, the correction concerns the nonlinear regime, and, thus, the linearized form of the new equations coincides with the linearized Navier–Stokes equations. Second, the solution (8.178) for Maxwell molecules and the result of the approximation (8.179) for other models suggest that the modified viscosity μR gives a vanishing contribution in the limit of very high values of the average velocity. This feature seems to be of no surprise: if the average velocity is very high in comparison to other characteristic velocities (in our case, to the heat velocity), no mechanisms of momentum transfer are relevant except for the transfer with the stream.

However, a cautious remark is in order since the original "kinetic" description are Grad's equations (8.166) and not the Boltzmann equation.

(iii) The invariance equation (8.176) defines the relevant physical solution to the viscosity factor for all values of g, and demonstrates an interesting phase-space behavior similar to those of finite-dimensional dynamical systems.

8.5 The Main Lesson

Up to now, the problem of the exact relationship between kinetics and hydrodynamics remains unsolved. All the methods used to establish this relationship are not rigorous, and involve approximations. In this work, we have considered situations where hydrodynamics is the exact consequence of kinetics, and in that respect, a new class of exactly solvable models of statistical physics has been established.

The main lesson we can learn from the exact solution is the following: The Chapman–Enskog method is the Taylor series expansion approach to solving the appropriate invariance equation. Alternative iteration methods are much more robust for solving this equation. Therefore, it seems quite important to develop approaches to the problem of reduced description based on the principle of dynamic invariance rather than on particular methods of solving the invariance equations. The exact solutions where these questions can be answered in all the quantitative details provide a sound motivation for such developments.