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Modified Kirchhoff Flow with a Partially Penetrable Obstacle and Its Application to the Efficiency of Free Flow Turbines

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Abstract—An explicitly solvable analog of the Kirchhoff flow for the case of a semipenetrable obstacle is considered. Its application to estimating the efficiency of free flow turbines is discussed.

Keywords—Cavitation flows, Kirchhoff method.

1. INTRODUCTION

The modification of the Kirchhoff model for the case of a partially penetrable obstacle proposed in this paper arose from the theoretical investigation of the efficiency of recently developed free flow hydraulic turbines, i.e., the turbines that work without dams [1]. For this kind of turbine, the term "efficiency" is referred to as the ratio of the power utilized by the turbine to the power carried by undisturbed flow through the projected area of the turbine to the plane perpendicular to the flow. For the propeller type turbine in free flows, the efficiency usually ranges from 10% to 20%. The best results are achieved by the Helical turbine that exhibited 35% efficiency in the well-documented tests [2]. Further development of turbines of that type makes very important the theoretical investigation of their efficiency limit.

Clearly, for a free flow turbine, the main problem is that any attempt to use the flow passing through the turbine more effectively would increase the streamlining part and might eventually decrease the net efficiency.

Therefore, in the first approximation, the turbine (or more generally, the entire section of turbines) can be substituted by a partially penetrable body absorbing the energy from the flow. In the case of ideal inviscid and incompressible liquid, using this model will encounter the well-known d'Alembert paradox, that the liquid meets no resistance from the streamlined body. In

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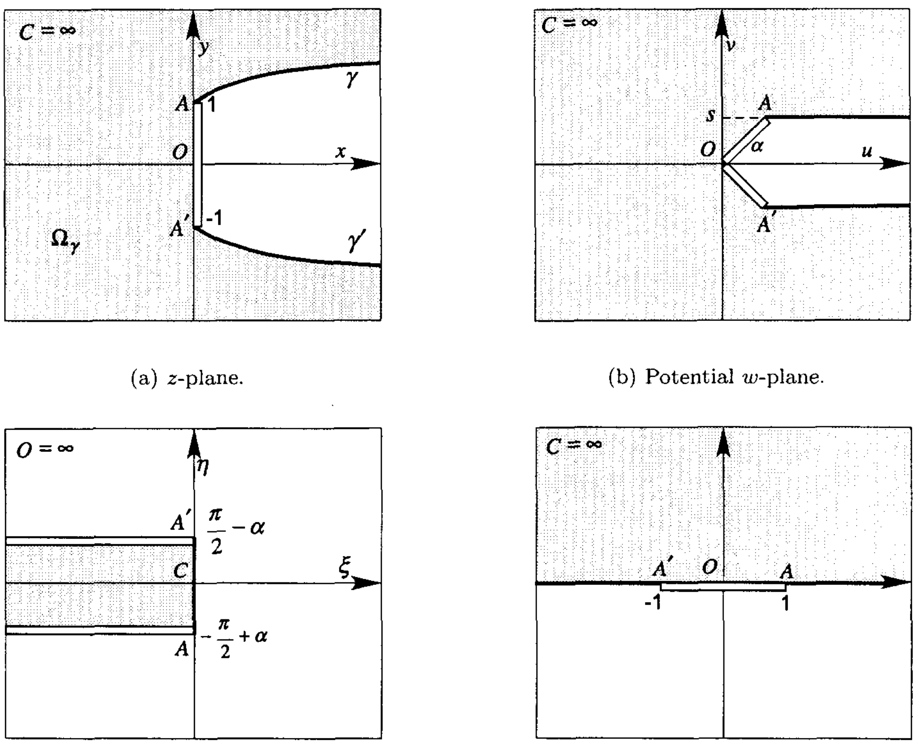
the case of streamlining without penetration, this paradox can be avoided by considering a discontinuous flow with separation. The classical example of this situation is the Kirchhoff flow that describes the direct impact of the stream on a lamina. In this case, the stream separates from its edges forming the stagnation domain past the lamina that leads to a nonzero drag [3].

In this paper, an analog of the Kirchhoff flow is constructed for the case of a semipenetrable lamina. In this case, if the flow crosses the lamina at the same angle α at any point, the problem can also be solved explicitly by the Kirchhoff method [3].

2. MODIFIED KIRCHHOFF FLOW

The Agrand diagram [3] of the modified Kirchhoff flow is shown in Figure 1. The current moves towards the positive direction of the x-axis and separates from the edges of the lamina as in the classical situation. Denote the separating streamlines by γ and γ'. For the sake of convenience, assume that the velocity at infinity and the half of the breadth of the lamina are both equal to one. (It can be done, since the efficiency we are looking for is dimensionless.) As in the classical case, the pressure past the lamina is assumed to be equal to the pressure at infinity, therefore, the velocity on the free streamlines is equal to the velocity at infinity.



(a) z-plane. (b) Potential w-plane.

(c) Hodograph ζ-plane. (d) t-plane.

Figure 1.

In the domain Ω_γ (see Figure 1a) the flow is laminar and irrotational, therefore, it can be described in terms of complex potential w, i.e., a holomorphic function, s.t.

$$\vec{V} = \overline{\frac{dw}{dz}}. \tag{1}$$

The potential maps Ω_γ onto the shape shown in Figure 1b. Now consider the hodograph plane

$$\zeta = \xi + i\eta = \log \frac{dw}{dz} = \log V - i \arg \vec{V}. \tag{2}$$

As in the classical case [3], the image of Ω_γ on the hodograph is a semistrip

$$S = \left\{(\xi,\eta) : \infty < \xi < 0,\ -\frac{\pi}{2} + \alpha < \eta < \frac{\pi}{2} - \alpha\right\}, \tag{3}$$

but in this case, it has the width of $2 - 4\alpha/\pi$, since the current crosses the lamina at the angle α.

As in the classical situation, the solution, i.e., the free boundaries γ and γ' and the potential w can be found by using the Kirchhoff method. The idea of this method can be described as follows. Since the images of Ω_γ on the w- and ζ-planes contain no free boundaries, their conformal representations can be easily found. Since $\zeta = \log \frac{dw}{dz}$, these conformal mappings yield the differential equation for w, and the solution is obtained by integrating this equation. Let the upper semiplane $\{t > 0\}$ be the canonical domain (Figure 1d). Then the conformal representation of the image of Ω_γ on the w-plane is constructed using the Christoffel-Schwarz integral

$$\frac{dw}{dt} = \frac{2s}{\alpha}\left(t^2 - 1\right)^{\alpha/\pi} t^{(1-2\alpha/\pi)}, \tag{4}$$

$$w(t) = \frac{2s}{\alpha}\int_0^t \left(\tau^2 - 1\right)^{\alpha/\pi} \tau^{(1-2\alpha/\pi)}\, d\tau. \tag{5}$$

Here, s is the width of the gap on w-plane, that can also be interpreted as the distance between free streamlines at infinity or the fraction of the flow that goes through the obstacle. The conformal representation of the semistrip S with the boundary extension as shown in Figure 1c is given by

$$\zeta = -\left(1 - \frac{2\alpha}{\pi}\right)\log\left(\frac{1}{t} + \frac{1}{t}\sqrt{1-t^2}\right) - i\left(\frac{\pi}{2} - \alpha\right). \tag{6}$$

Then

$$\log\frac{dz}{dw} = \left(1 - \frac{2\alpha}{\pi}\right)\log\left(\frac{1}{t} + \frac{1}{t}\sqrt{1-t^2}\right) + i\left(\frac{\pi}{2} - \alpha\right), \tag{7}$$

$$\frac{dz}{dw} = e^{i(\pi/2-\alpha)}\left(1 + \sqrt{1-t^2}\right)^{1-2\alpha/\pi} t^{2\alpha/\pi - 1}, \tag{8}$$

$$\frac{dz}{dt} = \frac{dz}{dw}\cdot\frac{dw}{dt} = \frac{2is}{\alpha}\left(1 + \sqrt{1-t^2}\right)^{1-2\alpha/\pi}\left(1 - t^2\right)^{\alpha/\pi}. \tag{9}$$

Since $\int_0^1 \frac{dz}{dt}\, dt = i$, then

$$s = \frac{\alpha}{2I_2}, \tag{10}$$

where

$$I_2 = \int_0^1 \left(1 + \sqrt{1-t^2}\right)^{1-2\alpha/\pi}\left(1 - t^2\right)^{\alpha/\pi} dt. \tag{11}$$

Then the formula

$$z(t) = \frac{i}{I_2}\int_0^t \left(1 + \sqrt{1-\tau^2}\right)^{1-2\alpha/\pi}\left(1 - \tau^2\right) d\tau \tag{12}$$

gives the conformal representation of the flow domain Ω_γ.

3. THE EFFICIENCY

In the introduction, the efficiency was defined as the ratio of the absorbed power to the power carried by undisturbed flow through the projected area of the obstacle perpendicular to the flow. In this model, the power absorbed by the lamina is given by

$$P = \int_{-1}^{1} [p] V_x\, dy, \tag{13}$$

where $[p]$ denotes the pressure drop across the lamina and V_x is the x-component of the velocity $\vec{V}$. The power carried by the undisturbed flow through the lamina of width 2 is

$$P_\infty = 2 \cdot \frac{\rho V_\infty^3}{2} = 1 \tag{14}$$

since ρ and V_∞ are both equal to 1. By virtue of the Bernoulli theorem, $[p] = (V_\infty^2 - V^2)/2 = (1 - V^2)/2$. Then

$$\mathcal{E} = \frac{P}{P_\infty} = \frac{1}{2}\int_{-1}^{1} V_x \left(1 - V^2\right) dy = \int_0^1 V_x \left(1 - V^2\right) dy \tag{15}$$

$$= \int_0^1 \left(\operatorname{Re}\frac{dw}{dz}\right)\left(1 - \left|\frac{dw}{dz}\right|^2\right) dy = \frac{1}{i}\int_0^1 \left(\operatorname{Re}\frac{dw}{dz}\right)\left(1 - \left|\frac{dw}{dz}\right|^2\right)\frac{dz}{dt}\, dt \tag{16}$$

$$= s - \frac{1}{i}\int_0^1 \left(\operatorname{Re}\frac{dw}{dz}\right)\left|\frac{dw}{dz}\right|^2 \frac{dz}{dt}\, dt \tag{17}$$

$$= s - \sin\alpha \int_0^1 \left|\frac{dw}{dz}\right|^3 \frac{dz}{i\,dt}\, dt = \frac{1}{I_2}\left(\frac{\alpha}{2} - I_3 \sin\alpha\right), \tag{18}$$

where

$$I_3 = \frac{I_2(\alpha)}{i}\int_0^1 \left|\frac{dw}{dz}\right|^3 \frac{dz}{dt}\, dt = \int_0^1 \left(1 + \sqrt{1 - t^2}\right)^{4\alpha/\pi - 2} \left(1 - t^2\right)^{\alpha/\pi} t^{3 - 6\alpha/\pi}\, dt. \tag{19}$$

The results of the numeric evaluation of the efficiency $\mathcal{E}$ and parameter s are given in Table 1. The inclination angle α ranges from 0 (no penetration) to $\pi/2$ (undisturbed flow). The maximum efficiency of 30% is attained when $\alpha = 3\pi/8$ and $s = 0.61$.

Table 1.

No.	Inclination Angle, α	Efficiency, $\mathcal{E}$	Flow Through, s
1	0.07854	0.01761	0.02294
2	0.23562	0.06922	0.09168
3	0.39270	0.09998	0.13559
4	0.54978	0.14717	0.20623
5	0.70686	0.19625	0.28793
6	0.86394	0.24371	0.38199
7	1.02102	0.28292	0.48983
8	1.17810	0.30113	0.61302
9	1.33518	0.27274	0.75331
10	1.49226	0.14158	0.91259

4. DISCUSSION AND CONCLUSIONS

1. An explicitly solvable situation of the Helmholtz-type cavitational flow is obtained for the case of a partially penetrable obstacle.
2. If this explicit solution is considered as the model of a section of free flow turbines, it gives the efficiency of 30%.
3. In this case, if the pressure past the lamina is equal to the pressure at infinity, the following simple estimate shows that the efficiency cannot exceed 38%. Let s denote the fraction of the flow that passes through the obstacle. Assume for simplicity that the velocity is perpendicular to the obstacle and is the same at any point and the velocity at infinity, projected area of the obstacle and the density of the liquid are all equal to 1. In that

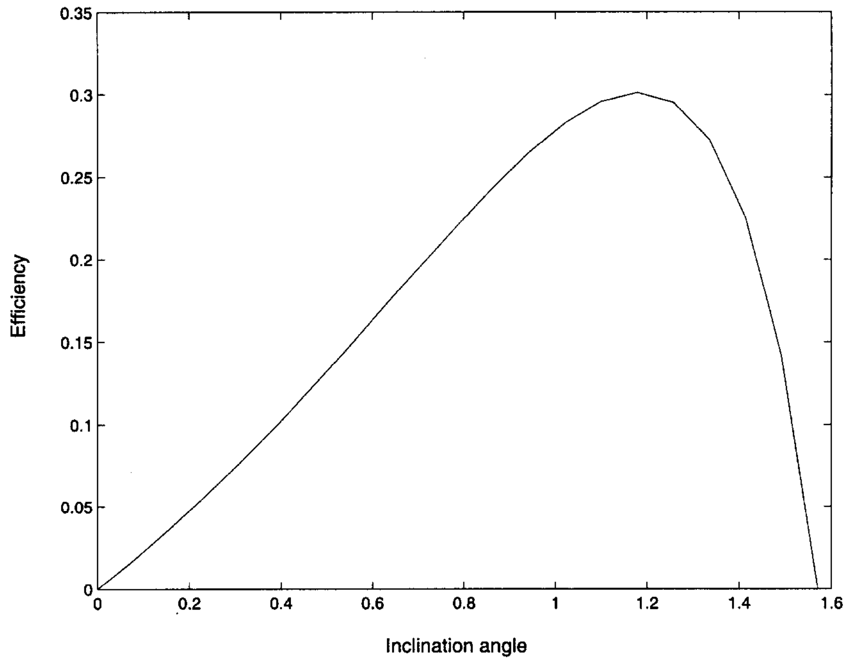


Figure 2. Efficiency $\mathcal{E}$ versus the inclination angle α.

case, the velocity through the obstacle is equal to s. Then pressure drop is $(1/2)(1 - s^2)$ and the absorbed power is $(1/2)(s - s^3)$ while the power carried by the undisturbed flow is $1/2$ and the efficiency is $s - s^3$. This function attains its maximum of $2/3\sqrt{3} \approx 0.38$ at $s = 1/\sqrt{3}$. In practice, the pressure past the turbine may be different, that requires a more accurate calculation.

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